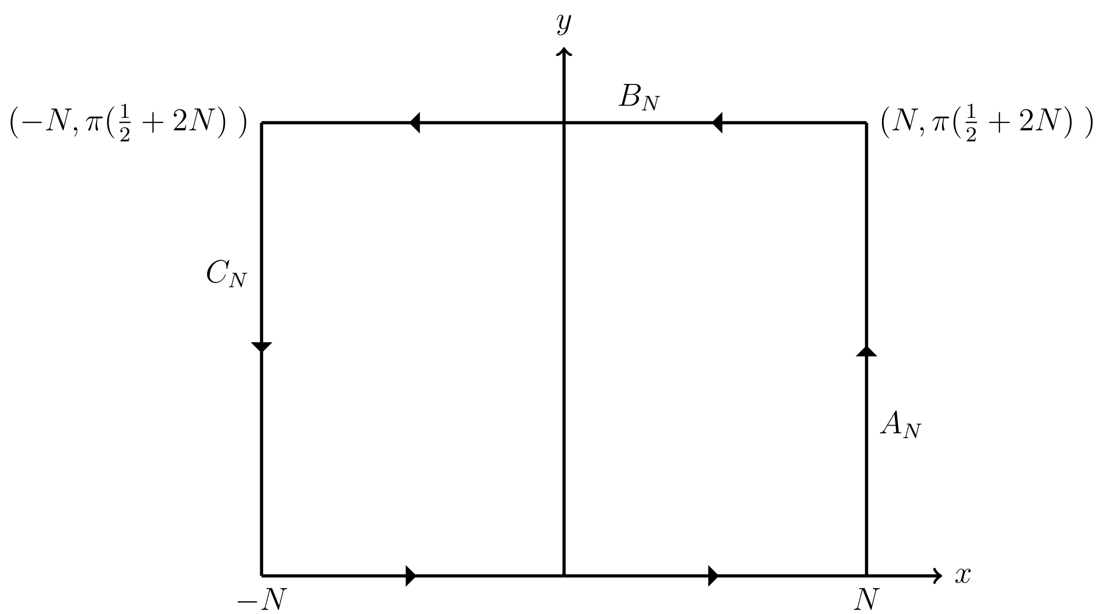
MATH 202 Complex Analysis
Homework 2
Solution Key

1) Evaluate the integral $\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right) \frac{dx}{x}$.

Solution:
Consider the function $f(z) = \frac{1}{z^2} - \frac{1}{z \sinh z}$ and the following path γ_N where $N > 0$ is an integer.



$$\gamma_N = [-N, N] + A_N + B_N + C_N$$

The residue theory says that

$$\int_{\gamma_N} f(z)\, dz = (2\pi i)(\text{sum of the residues of } f \text{ inside the path.})$$

Before starting our analysis we refresh our minds about some identities.

$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$
$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$
$|\sinh(x + iy)|^2 = \sinh^2 x + \sin^2 y \geq \sinh^2 x$, or simply $|\sinh(x + iy)| \geq \sinh x$
$|\cosh(x + iy)|^2 = \sinh^2 x + \cos^2 y \geq \sinh^2 x$, or simply $|\cosh(x + iy)| \geq \sinh x$.

Now we start calculating the residues of f.

First we observe that $z = 0$ is a removable singularity for $f(z) = \frac{1}{z^2} - \frac{1}{z \sinh z} = \frac{\sinh z - z}{z^2 \sinh z}$. Since

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots ,$$

we have

$$f(z) = \frac{\sinh z - z}{z^2 \sinh z} = \frac{\frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^3 + \frac{z^5}{3!} + \frac{z^7}{5!} + \cdots} = \frac{\frac{1}{3!} + \frac{z^2}{5!} + \cdots}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots},$$

showing that f can be defined at 0 as $\frac{1}{6}$. Hence there is no singularity at $z = 0$.

Hence we are interested only with the zeros of $\sinh(x + iy)$ with $0 < y < \pi(\frac{1}{2} + 2N)$.

Using the identities given at the beginning we find that the only poles of f inside γ_N are

$$z_n = n\pi i, n = 1, 2, \ldots, 2N.$$

Writing

$$f(z) = \frac{\frac{\sinh z - z}{z^2}}{\sinh z}$$

we see that

$$\mathrm{Res}_{z=z_n} f(z) = \left.\frac{\frac{\sinh z - z}{z^2}}{\cosh z}\right|_{z=z_n} = \frac{i}{\pi} \frac{(-1)^n}{n}.$$

Thus we get

$$(2\pi i)(\text{sum of the residues of } f \text{ inside the path.}) = 2\left[-\sum_{n=1}^{2N} \frac{(-1)^n}{n}\right].$$

Now we examine the integral of f along the parts of γ_N.

• On $[-N, N]$: $z = x$ and

$$\int_{[-N,N]} f(z)\, dz = 2\int_0^N f(x)\, dx,$$

since $f(x)$ is an even function.

• On A_N and C_N: Let L_N denote A_N or C_N.
On L_N we have $z = \pm N + it$, where $0 \le t \le \pi(\frac{1}{2} + 2N)$, and

$$|z| \ge N, \; |L_N| = \pi(\frac{1}{2} + 2N).$$

Moreover, using the identities given at the beginning we have

$$|\sinh z| \ge \sinh N \ge N, \text{ and hence } |z \sinh z| \ge N^2.$$

Thus

$$\left|\int_{L_N} f(z)\, dz\right| \le \left|\int_{L_N} \frac{dz}{z^2}\right| + \left|\int_{L_N} \frac{dz}{z \sinh z}\right| \le 2\frac{\pi(\frac{1}{2} + 2N)}{N^2} \to 0 \text{ as } N \to \infty.$$

• On B_N: Here $z = x + i\pi(\frac{1}{2} + 2N)$, where $-N \le x \le N$. In particular $\sinh(x + i\pi(\frac{1}{2} + 2N)) = \cosh x$. Thus $|z| \ge N$ and $|z \sinh z| \ge N \cosh x$. We then have

$$\begin{aligned} \left|\int_{B_N} f(z)\, dz\right| &\le \left|\int_{B_N} \frac{dz}{z^2}\right| + \left|\int_{B_N} \frac{dz}{z \sinh z}\right| \\ &\le \frac{2N}{N^2} + \frac{1}{N}\int_{-N}^{N} \frac{dx}{\cosh x} \\ &= \frac{2}{N} + \frac{1}{N}\left(2\arctan(e^x)\Big|_{-N}^{N}\right). \end{aligned}$$

The last equality can be verified easily by taking the derivative of $\arctan(e^x)$. Moreover we have

$$\lim_{N\to\infty} \arctan(e^N) = \frac{\pi}{2} \text{ and } \lim_{N\to\infty} \arctan(e^{-N}) = 0.$$

This shows that

$$\lim_{N\to\infty} \int_{B_N} f(z)\, dz = 0.$$

Putting these together and taking the limit as $N \to \infty$ we get

$$2\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} = 2\left[-\sum_{n=1}^\infty \frac{(-1)^n}{n}\right].$$

We recognize the above infinite sum as $\ln 2$, and finally get

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} = \ln 2.$$

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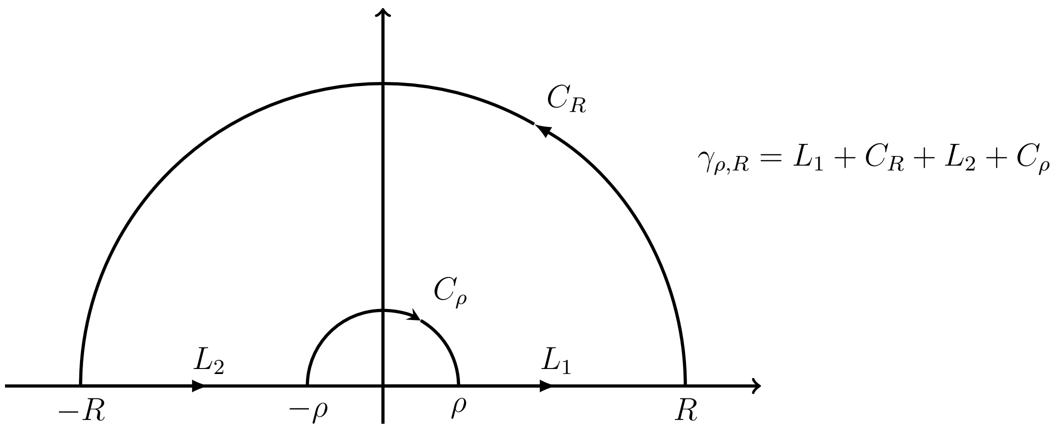
2) Evaluate the integral $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\, dx$, where $a, b \geq 0$.

Solution:

Use the contour $\gamma_{\rho,R}$ given below with the function $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$.



f has no poles inside the contour so we have

$$\int_{\gamma_{\rho,R}} f(z)\, dz = 0.$$

We then examine the Laurent expansion of f at $z = 0$.

$$f(z) = \frac{(1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \cdots) - (1 + \frac{ibz}{1!} + \frac{(ibz)^2}{2!} + \frac{(ibz)^3}{3!} + \cdots)}{z^2}$$

$$= \frac{i(a-b)}{z} + \frac{(ia)^2 - (ib)^2}{2!} + \frac{(ia)^3 - (ib)^3}{3!}\, z + \cdots .$$

Thus $z = 0$ is a simple pole of f with residue $i(a - b)$.

• On C_ρ: The above information immediately gives, via the *useful lemma*, that

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\, dz = (-\pi i)(i(a-b)) = \pi(a-b).$$

We then examine the behavior of the integral along the other portions of the contour.

• On C_R: Here $|z| = R$, and $z = x + iy$ with $y \geq 0$. Then

$$|f(z)| \leq \frac{\left|e^{iaz}\right| + \left|e^{ibz}\right|}{|z|^2} = \frac{e^{-ay} + e^{-by}}{R^2} \leq \frac{2}{R^2},$$

and hence

$$\left|\int_{C_R} f(z)\, dz\right| \leq \frac{2}{R^2}\, \pi R \to 0 \;\text{ as }\; R \to \infty.$$

• On L_1: Here $z = x$ with $\rho \leq x \leq R$.

$$\int_{L_1} f(z)\, dz = \int_{\rho}^{R} \frac{e^{iax} - e^{ibx}}{x^2}\, dx.$$

• On $-L_2$: Here $z = -x$ with $\rho \leq x \leq R$. Then $dz = -dx$.

$$\int_{L_2} f(z)\, dz = -\int_{-L_2} f(z)\, dz = \int_{\rho}^{R} \frac{e^{-iax} - e^{-ibx}}{x^2}\, dx.$$

Thus we get

$$\int_{L_1} f(z)\, dz + \int_{L_2} f(z)\, dz = 2\int_{\rho}^{R} \frac{\cos ax - \cos bx}{x^2}\, dx.$$

Putting these in and taking the limit as $\rho \to 0$ and $R \to \infty$ we get

$$2\int_{0}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx + \pi(b - a) = 0,$$

giving us finally

$$\int_{0}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx = \frac{\pi}{2}(b - a),\ \ a, b \geq 0.$$

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3) Evaluate the integral $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$.

Solution:

An acceptable and a very easy solution to this problem is to use the result of the previous problem.

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\, dx = \frac{\pi}{2}(b-a),\ \ a, b \geq 0.$$

Here put $a = 0$ and $b = 2$ to get

$$\int_0^\infty \frac{1-\cos 2x}{x^2}\, dx = 2\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \pi,$$

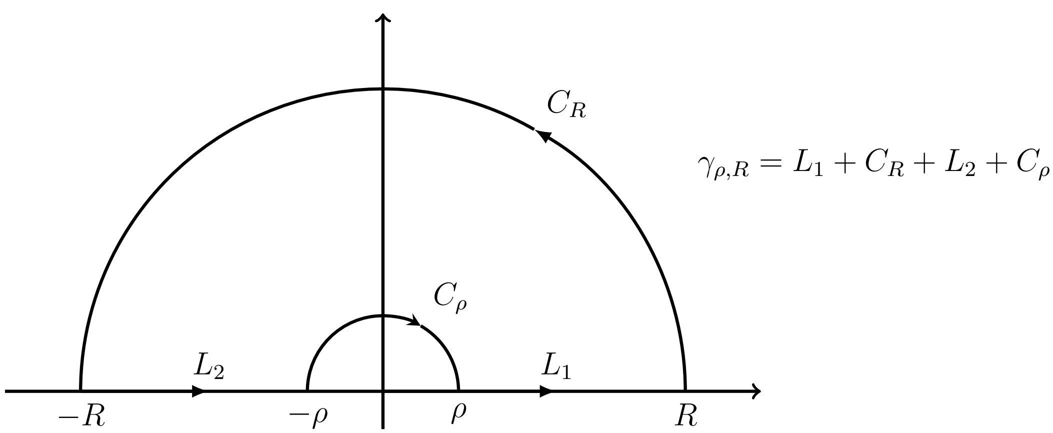
and hence the result

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

However we may want to do this the hard way. Then we recall the half angle formula that $2\sin^2 x = 1 - \cos 2x$ and decide to use the function

$$f(z) = \frac{1 - e^{i2z}}{z^2}$$

on the contour



We first observe that

$$f(z) = \frac{-2i}{z} + 2 + \frac{4i}{3}z - \frac{2}{3}z^2 + \cdots,$$

so $f(z)$ has a simple pole at $z = 0$ with residue $B_0 = -2i$.

• On C_ρ: By the *useful lemma* we see that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\, dz = -\pi i B_0 = -\pi i(-2i) = -2\pi.$$

• On C_R: $z = Re^{i\theta} = R\cos\theta + i\sin\theta$, $0 \leq \theta \leq \pi$, $dz = Rie^{i\theta}d\theta$. Then

$$\left|\int_{C_R} f(z)\, dz\right| = \left|\int_0^\pi \frac{1 - e^{iR\cos\theta}e^{-R\sin\theta}}{R^2e^{i2\theta}} Rie^{i\theta}\, d\theta\right| \leq \frac{\pi}{R} + \frac{1}{R}\int_0^\pi e^{-R\sin\theta}\, d\theta \to 0 \ \text{ as } \ R \to \infty.$$

• On L_1: $z = x$, $\rho \le x \le R$, and

$$\int_{L_1} f(z)\, dz = \int_\rho^R \frac{1 - e^{i2x}}{x^2}\, dx = 2\int_\rho^R \frac{\sin^2 x}{x^2}\, dx - i\int_\rho^R \frac{\sin 2x}{x^2}\, dx.$$

• On $-L_2$: $z = -x$, $\rho \le x \le R$, $dz = -dx$, and

$$\int_{L_2} f(z)\, dz = -\int_{-L_2} f(z)\, dz = \int_\rho^R \frac{1 - e^{-i2x}}{x^2}\, dx = 2\int_\rho^R \frac{\sin^2 x}{x^2}\, dx + i\int_\rho^R \frac{\sin 2x}{x^2}\, dx.$$

Putting these together and taking the limits as $\rho \to 0$ and $R \to \infty$ we get

$$4\int_0^\infty \frac{\sin^2 x}{x^2}\, dx - 2\pi = 0,$$

or

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

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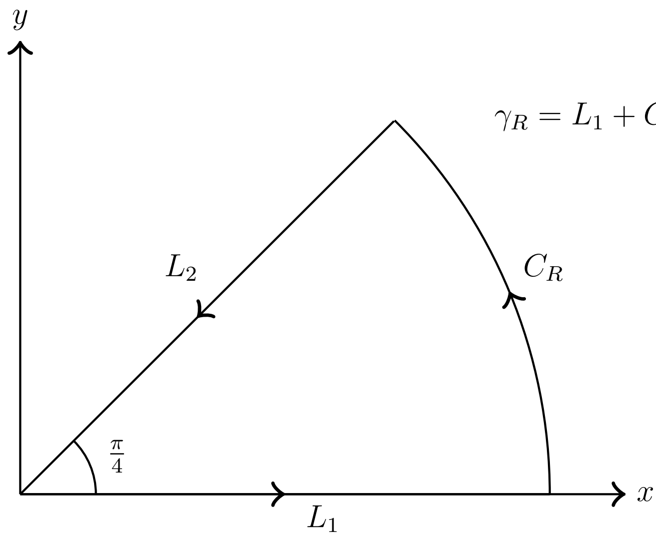
4) Evaluate the integral $\int_0^\infty \sin x^2 \, dx$.

Solution:

Use the function $f(z) = e^{iz^2}$ together with the contour given below.



There are no poles of $f(z)$ inside the contour so we have

$$\int_{\gamma_R} f(z) \, dz = 0.$$

• On L_1: $z = x$, $0 \le x \le R$ and

$$\int_{L_1} f(z) \, dz = \int_0^R e^{ix^2} dx = \int_0^R \cos x^2 \, dx + i \int_0^R \sin x^2 \, dx.$$

• On $-L_2$: $z = \alpha x$, $0 \le x \le R$ and $dz = \alpha dx$ where $\alpha = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. Note that $\alpha^2 = i$ so $z^2 = ix^2$.
Then we have

$$\int_{L_2} f(z) \, dz = -\int_{-L_2} f(z) \, dz = -\alpha \int_0^R e^{-x^2} \, dx \to -\alpha\frac{\sqrt{\pi}}{2} \text{ as } R \to \infty \quad \text{(from Calculus)}$$

• On C_R: $z = Re^{i\theta}$, $0 \le \theta \le \pi/4$, $dz = Rie^{i\theta}d\theta$, $z^2 = R^2e^{i2\theta} = R^2\cos 2\theta + iR^2 \sin 2\theta$. Then

$$\left|\int_{C_R} f(z) \, dz\right| = \left|\int_0^{\pi/4} e^{iR^2\cos 2\theta} e^{-R^2 \sin 2\theta} iRe^{i\theta} d\theta\right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin t} dt \le \frac{R}{2}\frac{\pi}{2R^2} \to 0 \text{ as } R \to \infty.$$

Putting these together and taking the limit as $R \to \infty$ we get

$$\int_0^\infty \cos x^2 \, dx + i \int_0^\infty \sin x^2 \, dx - \alpha\frac{\sqrt{\pi}}{2} = 0.$$

Equating real and imaginary parts separately we finally obtain

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$