Date: 8 January 2022 Saturday
Time: 12:00-14:30
Instructor: Ali Sinan Sertöz

NAME:..

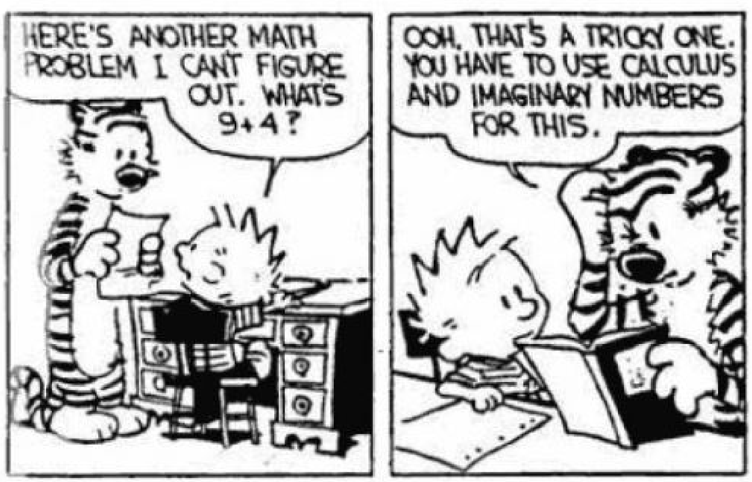
STUDENT NO:..

Math 202 Complex Analysis – Final Exam Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.



Calvin and Hobbes cartoon by William B. Watterson

NAME: STUDENT NO:

Q-1) Calculate the principal value of $i^{(1+i)}$ and write the result in the rectangular form $a + ib$ where $a, b \in \mathbb{R}$. *(Recall that "principal value" means that the argument of a complex number is to be considered between $-\pi$ and π.)*

Solution:

$$\begin{aligned} i^{(1+i)} &= \exp[(1+i)\log i] \\ &= \exp[(1+i)(i\frac{\pi}{2})] \\ &= \exp[-\frac{\pi}{2} + i\frac{\pi}{2}] \\ &= e^{-\pi/2}[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \\ &= ie^{-\pi/2} \end{aligned}$$

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Q-2) Consider the mapping given by

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Describe the images, under this mapping, of the circles $|z| = R > 0$. What happens when $R = 1$?

Solution:

Let $z = x + iy$ and $w = u + iv$.

Then we have

$$w = \frac{1}{2}\left(x + \frac{x}{x^2 + y^2}\right) + \frac{i}{2}\left(y - \frac{y}{x^2 + y^2}\right)$$

and hence when $x = R\cos\theta$ and $y = R\sin\theta$, we have

$$u = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta, \text{ and } v = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta.$$

Eliminating θ between u and v we get, when $R \neq 1$,

$$\frac{u^2}{\frac{1}{4}\left(R + \frac{1}{R}\right)^2} + \frac{v^2}{\frac{1}{4}\left(R - \frac{1}{R}\right)^2} = 1.$$

Thus the images of the circle $|z| = R$ under this map are ellipses.

Note however that for $R > 0$, the circles with radii R and $1/R$ map to the same ellipse but with reverse orientation. This is because they lie on different copies of the w-plane. These two planes are cut along $[-1, 1]$ and glued together, the upper part of one to the lower part of the other.

When $R = 1$, the unit circle in the z-plane is then mapped onto this cut, the upper part of the circle being in one sheet and the lower part in the other sheet.

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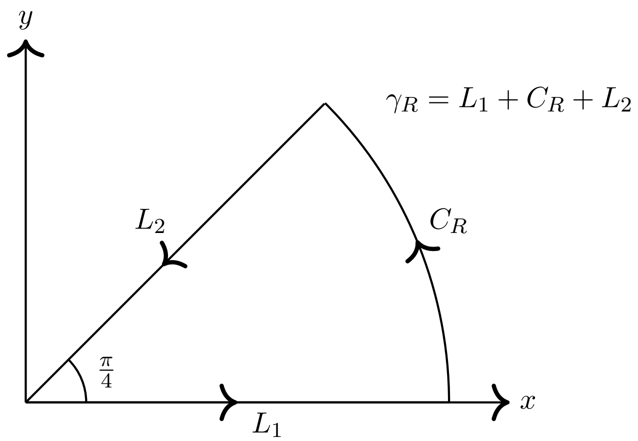
Q-3) Evaluate the integral $\displaystyle\int_0^\infty \cos x^2 \, dx$.

Hint: You may find the following contour useful.



Solution:

Use the function $f(z) = e^{iz^2}$ together with the contour given above.

There are no poles of $f(z)$ inside the contour so we have

$$\int_{\gamma_R} f(z) \, dz = 0.$$

• On L_1: $z = x$, $0 \le x \le R$ and

$$\int_{L_1} f(z) \, dz = \int_0^R e^{ix^2} dx = \int_0^R \cos x^2 \, dx + i \int_0^R \sin x^2 \, dx.$$

• On $-L_2$: $z = \alpha x$, $0 \le x \le R$ and $dz = \alpha dx$ where $\alpha = \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}$. Note that $\alpha^2 = i$ so $z^2 = ix^2$. Then we have

$$\int_{L_2} f(z) \, dz = -\int_{-L_2} f(z) \, dz = -\alpha \int_0^R e^{-x^2} \, dx \to -\alpha \frac{\sqrt{\pi}}{2} \text{ as } R \to \infty \quad \text{(from Calculus)}$$

• On C_R: $z = Re^{i\theta}$, $0 \le \theta \le \pi/4$, $dz = Rie^{i\theta} d\theta$, $z^2 = R^2 e^{i2\theta} = R^2 \cos 2\theta + iR^2 \sin 2\theta$. Then

$$\left| \int_{C_R} f(z) \, dz \right| = \left| \int_0^{\pi/4} e^{iR^2 \cos 2\theta} e^{-R^2 \sin 2\theta} iRe^{i\theta} d\theta \right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin t} dt \le \frac{R}{2} \frac{\pi}{2R^2} \to 0 \text{ as } R \to \infty.$$

Putting these together and taking the limit as $R \to \infty$ we get

$$\int_0^\infty \cos x^2 \, dx + i \int_0^\infty \sin x^2 \, dx - \alpha \frac{\sqrt{\pi}}{2} = 0.$$

Equating real and imaginary parts separately we finally obtain

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

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Q-4) Prove that $\prod_{k=0}^{\infty}\left(1+z^{2^k}\right)$ converges uniformly on compact subsets of $|z|<1$. Find the limiting function.

(This is exercise 6 at the end of chapter 17 in Bak & Newman, Complex Analysis.)
Hint: A power series converges uniformly on compact subsets of its domain of convergence.

Solution:

Let $P_n = \prod_{k=0}^{n}\left(1+z^{2^k}\right)$ be the sequence of partial products.

It is immediate to see, after writing down the first few terms of the sequence, that

$$P_n = 1 + z + z^2 + \cdots + z^{2^{n+2}-1}.$$

Hence

$$\prod_{k=0}^{\infty}\left(1+z^{2^k}\right) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z},$$

where we know that the sum converges uniformly on compact subsets of $|z|<1$.

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Q-5) Show that the polynomial

$$P(z) = (2022)^2\, z^{2022} + \sum_{k=1}^{2021} k z^k + 2022$$

has all its roots inside $|z| < 1$.

Hint: You may find Rouché's theorem useful: (verbatim from Bak & Newman) Supoose that f and g are analytic inside and on a regular closed curve γ and that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then $f + g$ and f have the same number of zeros inside γ.

Solution:

Let

$$f(z) = (2022)^2 z^{2022} \text{ and } g(z) = \sum_{k=1}^{2021} k z^k + 2022.$$

Define the real valued function $G(z)$ as

$$G(z) = 2021|z|^{2021} + 2020|z|^{2020} + \cdots + |z| + 2022.$$

By triangle inequality we know that

$$|g(z)| \leq G(z).$$

It is also clear that

$$\frac{|f(z)|}{|g(z)|} \geq \frac{|f(z)|}{G(z)}.$$

Now putting $|z| = 1$ we get

$$\frac{|f(z)|}{|g(z)|} \geq \frac{|f(z)|}{G(z)} = \frac{2022^2}{1 + 2 + \cdots + 2022} = \frac{2022 \cdot 2022}{1011 \cdot 2023} > 1.$$

Hence for $|z| = 1$ we have $|f(z)| > |g(z)|$. Then by Rouché's theorem $f(z)$ and $P(z) = f(z) + g(z)$ have the same number of zeros inside $|z| < 1$, counting multiplicities. Clearly $f(z)$ has 2022 zeros inside the unit disc and thus $P(z)$ has 2022 zeros there. But $P(z)$ is a polynomial of degree 2022, so it has no other zeros elsewhere.