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Math 206 Complex Calculus – Final Exam
Solutions

Q-1) Solve the following recursion equation:

$$f(n+2) - 7f(n+1) + 12f(n) = 2^n,\ f(0) = f(1) = 0.$$

Solution: Using $\mathcal{Z}$-transformation we recall that

$$\begin{aligned}
\mathcal{Z}(f(n)) &= F(z),\\
\mathcal{Z}(f(n+1) &= zF(z) - zf(0) = zF(z),\\
\mathcal{Z}(f(n+2) &= z^2F(z) - z^2f(0) - zf(1) = z^2F(z),\\
\mathcal{Z}(2^n) &= \frac{z}{z-2}.
\end{aligned}$$

Taking the $\mathcal{Z}$-transform of both sides of the equation we get

$$\begin{aligned}
(z^2 - 7z + 12)F(z) &= \frac{z}{z-2}, \quad \text{or}\\
F(z) &= \frac{z}{(z-2)(z-3)(z-4)}.
\end{aligned}$$

Recalling that under the $\mathcal{Z}$-transform most functions go to a fraction with a z in the numerator, we use the partial fractions technique as follows;

$$\begin{aligned}
F(z) &= \frac{z}{(z-2)(z-3)(z-4)}\\
&= z\left[\frac{1}{(z-2)(z-3)(z-4)}\right]\\
&= z\left[\frac{1}{2}\frac{1}{z-2} - \frac{1}{z-3} + \frac{1}{2}\frac{1}{z-4}\right]\\
&= \frac{1}{2}\frac{z}{z-2} - \frac{z}{z-3} + \frac{1}{2}\frac{z}{z-4}
\end{aligned}$$

Taking inverse $\mathcal{Z}$-transform now gives

$$f(n) = (\frac{1}{2})2^n - 3^n + (\frac{1}{2})4^n,$$

or after simplifying,

$$f(n) = 2^{n-1} + 2^{2n-1} - 3^n,\ n = 0, 1, ...$$

You should in the exam check that the answer you find is actually a solution of the given equation.

Q-2) Let R be the region defined as

$$R = \{\, z \in \mathbb{C} \mid 1 \le |z| \le 2,\ \operatorname{Im} z \ge 0 \,\}$$

Consider the transformation $f(z) = z + \dfrac{1}{z}$.
Describe $f(R)$.
Describe the image of the boundary of R.
Is the transformation conformal?

Solution: This map is studied on page 374 of your book.
Let $z = re^{i\theta}$. Then

$$f(z) = u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

If $r = 1$, then $w = 2\cos\theta$ and the inner circle maps onto the real interval $[-2, 2]$ in the w plane.
If $1 < r \le 2$, then $r - 1/r \ne 0$ and we obtain

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1.$$

If $0 \le \theta \le \pi$ and $r > 1$, then $v > 0$, so we get the part of this ellipse which is in the upper half plane. Putting $r = 2$ we find the outermost ellipse in the image. The interior points of R are mapped to the interior points of this outermost ellipse with $v > 0$.
The outer circle, $r = 2$, maps onto this outermost ellipse.
When $\theta = 0$, f maps $[1, 2]$ onto $[2, 5/2]$.
When $\theta = \pi$, f maps $[-2, -1]$ onto $[-5/2, -2]$.
$f'(z) = 1 - 1/z^2 = 0$ only at $z = \pm 1$, so f is conformal at every other point.

Q-3) Solve the following boundary value problem for a bounded T;

$$\begin{aligned} T_{xx}(x,y) + T_{yy}(x,y) &= 0, \quad y \ge 0,\ -\infty < x < \infty, \\ T(x,0) &= 0, \quad x < -2, \\ T(x,0) &= 1, \quad x > 2, \\ T_y(x,0) &= 0, \quad -2 < x < 2. \end{aligned}$$

Solution: This is *almost* Exercise 6 on page 308, and the solution uses exactly the same argument given on page 306.
Consider the region R given in the w plane by $v \ge 0$ and $-\pi/2 \le u \le \pi/2$. The map $z = 2\sin w$ sends this region onto our region, conformally except at the points $u = \pm\pi/2$. A solution to our problem in R is $T(u, v) = (1/2) + (1/\pi)u$. Check that it is a solution.
$z = 2\sin w$ becomes $x + iy = 2\sin u\cosh v + i2\cos u\sinh v$. Eliminating v we get

$$\frac{x^2}{4\sin^2 u} - \frac{y^2}{4\cos^2 u} = 1.$$

Using the properties of hyperbolas, this gives

$$4\sin u = \sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2}$$

and solving for u finally gives

$$T(x,y) = \frac{1}{2} + \frac{1}{\pi}\arcsin\left[\frac{1}{4}\left(\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2}\right)\right],$$

where $-\pi/2 \le \arcsin t \le \pi/2$ since this is the range for u.

Q-4) Describe the image of the x-axis under the Schwarz-Christoffel transformation

$$f(z) = \alpha \int_0^z (s^2 - 1)^{-3/4} \, s^{-1/2} ds, \quad \text{where } \alpha = e^{i3\pi/4}.$$

Hint: $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$, $p,q > 0$, is the Beta function and in particular $B(1/4, 1/4) = 7.416...$

Solution: This is a reformulation of Exercise 1 on page 336.
We can set $x_1 = -1$, $x_2 = 0$, $x_3 = 1$. The corresponding constants describing the angles are $k_1 = 3/4$, $k_2 = 1/2$, $k_3 = 3/4$. Since $k_1 + k_2 + k_3 = 2$, the image is a triangle. Since one of the angles is $k_2\pi = \pi/2$, this is a right triangle. Since $k_1 = k_3$, this is an isosceles right triangle. $f(0) = 0$ is the right angle vertex of the triangle. To find $f(1)$ we evaluate the integral:

$$f(1) = \alpha \int_0^1 (s^2 - 1)^{-3/4} \, s^{-1/2} ds,$$

but here the $(s^2 - 1)$ factor is negative and a fourth root of it will be imaginary. We write it as

$$\begin{aligned} (s^2 - 1)^{-3/4} &= (-1)^{-3/4}(1 - s^2)^{-3/4} \\ &= \alpha^{-1}(1 - s^2)^{-3/4} \end{aligned}$$

and the integral becomes

$$f(1) = \int_0^1 (1 - s^2)^{-3/4} \, s^{-1/2} ds$$

which is a real integral. Say $f(1) = b \in \mathbb{R}^+$. Writing the integral for $f(-1)$ and making the substitution $t = -s$ we obtain that $f(-1) = if(1) = ib$. Furthermore making the substitution $t = s^2$ in the integral for $f(1)$ we find that $b = (1/2)B(1/4, 1/4)$.
Thus the real line maps onto the isosceles right triangle with right vertex at the origin and the other vertices at $(b, 0)$ and $(0, ib)$.