Date: 16 March 2002, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 206 Complex Calculus – Midterm Exam I
Solutions

1 Calculate all the fourth roots of -1.

Solution: $-1 = e^{i\pi(2n+1)}$, $n \in \mathbb{N}$. Fourth roots of -1 correspond to $e^{i\pi(2n+1)/4}$ for $n = 0, 1, 2, 3$. These are:

$$\begin{aligned} n = 0; \qquad & e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}. \\ n = 0; \qquad & e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}. \\ n = 0; \qquad & e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}. \\ n = 0; \qquad & e^{i\frac{7\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}. \end{aligned}$$

---

2 Calculate all values of z_0^c where $z_0 = \sqrt{3} + i$ and $c = (\sqrt{2}e^{i\pi/4})^{-1}$. Indicate which value is principal.

Solution:

$$\begin{aligned} z_0 &= \sqrt{3} + i, \\ &= 2e^{i(\frac{\pi}{6} + 2n\pi)}, \; n \in \mathbb{N}. \\ c &= (\sqrt{2}e^{i\pi/4})^{-1}, \\ &= \frac{1}{2} - i\frac{1}{2}. \\ z_0^c &= exp(c \log z_0) \\ &= exp([\frac{1}{2} - i\frac{1}{2}][\ln 2 + i(\frac{\pi}{6} + 2n\pi]). \\ &= exp([\frac{1}{2}\ln 2 + \frac{\pi}{12} + n\pi] + i[\frac{\pi}{12} + n\pi - \frac{1}{2}\ln 2]). \end{aligned}$$

The principal value corresponds to $n = 0$, in which case

$$\begin{aligned} z_0^c &= exp([\frac{1}{2}\ln 2 + \frac{\pi}{12}] + i[\frac{\pi}{12} - \frac{1}{2}\ln 2]), \\ &= 1.83 - i0.15 \quad \text{(approximately)}. \end{aligned}$$

---

3) Evaluate the integral $\int_C z\cos(\pi z)\ dz$ where the path C is parametrized as $z(t) = t + it^3\sin(\pi/t)$ for $0 < t \leq 1$ and $z(0) = 0$.

Solution: The integrand is an entire function so its path integrals along smooth contours are independent of path, but depend only on the end points. The given parametrization describes a smooth path. So the given integral depends only on the end points.

$$\begin{aligned}\int_C z\cos(\pi z)\ dz &= \int_0^1 z\cos(\pi z)dz \\ &= [(\frac{z\sin(\pi z)}{\pi} + \frac{\cos(\pi z)}{\pi^2}|_0^1] \\ &= -\frac{2}{\pi^2}.\end{aligned}$$

---

4) Evaluate the integral $\int_{|z|=\frac{3}{2}} \frac{\cos(\pi z)}{(z-1)z^2(z-2)}dz.$

Solution: Let $g(z) = \frac{\cos(\pi z)}{z-2}$. Note that $g(z)$ is analytic in the region of integration. The integrand can then be expressed as $\frac{g(z)}{(z-1)z^2}$. Using partial fractions technique we can write $\frac{1}{(z-1)z^2} = \frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2}$, and the integrand becomes $\frac{g(z)}{(z-1)z^2} = \frac{g(z)}{z-1} - \frac{g(z)}{z} - \frac{g(z)}{z^2}$.
Using the Cauchy Integral Formula the integral is then equal to
$2\pi i\{g(1) - g(0) - g'(0)\} = \frac{7\pi}{2}i.$
