1) Find all values of \( z^c \) and show which one is the principal value, where \( z \) and \( c \) are given as:

(a) \( z = 2e^{i\frac{5\pi}{4}}, c = 1 + i \). (b) \( z = -1 - \sqrt{3}i, c = \sqrt{3}i \).

Solution: In general \( z^c = \exp(c \log z) \) and the principal value is obtained when you take the argument of \( z \) to lie between \(-\pi\) and \(\pi\).

Solution a:

\[
z^c = \exp((1 + i) \log(2e^{i(5\pi/4+2n\pi)})), \quad n \in \mathbb{Z}
\]

\[
= \exp((1 + i)(\ln 2 + i(5\pi/4 + 2n\pi)))
\]

\[
= \exp[\ln 2 - (\frac{5\pi}{4} + 2n\pi)](\cos(\ln 2 + \frac{5\pi}{4} + 2n\pi) + i \sin(\ln 2 + \frac{5\pi}{4} + 2n\pi))
\]

Principal value is obtained when \( n \) is such that \(-\pi < \frac{5\pi}{4} + 2n\pi < \pi\). This is satisfied for \( n = -1 \) and then the principal argument of \( z \) becomes \(-\frac{3\pi}{4}\).

Hence the principal value for \( z^c \) is

\[
\exp[\ln 2 + \frac{3\pi}{4}][\cos(\ln 2 - \frac{3\pi}{4}) + i \sin(\ln 2 - \frac{3\pi}{4})] \approx -1.9 - 21i.
\]

Solution b: \( z = -1 - \sqrt{3}i = 2e^{(-2\pi/3+2n\pi)}, n \in \mathbb{Z} \).

\[
z^c = \exp(\sqrt{3}i \log 2e^{(-2\pi/3+2n\pi)})
\]

\[
= \exp(\sqrt{3}i(\ln 2 + i(-2\pi/3 + 2n\pi)))
\]

\[
= \exp(-\sqrt{3}(-2\pi/3 + 2n\pi + i\sqrt{3}\ln 2))
\]

\[
= \exp(-\sqrt{3}(-2\pi/3 + 2n\pi))[\cos(\sqrt{3}\ln 2) + i \sin(\sqrt{3}\ln 2)]
\]

The principal value here corresponds to \( n = 0 \):

\[
z^c = \exp(2\sqrt{3}\pi/3)[\cos(\sqrt{3}\ln 2) + i \sin(\sqrt{3}\ln 2)] \approx 13.6 + 35i.
\]

(The numerical calculation was not required in the quiz.)