Date: 30 May, 2003, Friday
Instructor: Ali Sinan Sertöz
Time: 9:00-11:00

Math 206 Complex Calculus – Final Exam
Solutions

1 Solve the following differential equation using Laplace transform techniques:

$$f''(t) - 3f'(t) + 2f(t) = e^{3t}$$

where $f(0) = 0$, $f'(0) = 1$.

Solution: We apply Laplace transform to both sides of the differential equation using the formulas

$$\begin{aligned}
\mathfrak{L}(f(t)) &= F(s), \\
\mathfrak{L}(f'(t)) &= sF(s) - f(0) \\
&= sF(s), \\
\mathfrak{L}(f''(t)) &= s^2F(s) - sf(0) - f'(0) \\
&= s^2F(s) - 1, \\
\mathfrak{L}(e^{3t}) &= \frac{1}{s-3}.
\end{aligned}$$

The equation then becomes

$$(s^2 - 3s + 2)F(s) - 1 = \frac{1}{s-3}.$$

Note that $s^2 - 3s + 2 = (s-1)(s-2)$. Solving for $F(s)$ we find that

$$\begin{aligned}
F(s) &= \frac{1}{(s-1)(s-2)}\left(\frac{1}{s-3} - 1\right) \\
&= \frac{1}{(s-1)(s-2)}\left(\frac{s-2}{s-3}\right) \\
&= \frac{1}{(s-1)(s-3)} \\
&= -\frac{1}{2}\frac{1}{(s-1)} + \frac{1}{2}\frac{1}{(s-3)}.
\end{aligned}$$

We easily take Laplace inverse transform of both sides of this equation and get

$$f(t) = -\frac{1}{2}e^t + \frac{1}{2}e^{3t}.$$

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2 Calculate all values of $(-4)^{-3/2}$, and indicate the principal value.

Solution:

$$\begin{aligned}(-4)^{-3/2} &= \exp\left(-\frac{3}{2}\log(-4)\right)\\ &= \exp\left(-\frac{3}{2}[\ln 4 + i(2n+1)\pi]\right)\\ &= \exp\left(\ln\frac{1}{8} - i\frac{3}{2}(2n+1)\pi\right)\\ &= \frac{1}{8}\left(\cos\frac{3}{2}(2n+1)\pi - i\sin\frac{3}{2}(2n+1)\pi\right), \quad n \in \mathbb{Z}.\end{aligned}$$

The principal value is obtained when $n = 0$:

$$\begin{aligned}(-4)^{-3/2} &= \frac{1}{8}\left(\cos\frac{3}{2}\pi - i\sin\frac{3}{2}\pi\right)\\ &= \frac{i}{8}.\end{aligned}$$

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3) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^{1/5}}{1+x^5}dx.$

Solution:

We integrate the function $f(z) = \dfrac{z^{1/5}}{1+z^5} = \dfrac{\exp(\frac{1}{5}\log z)}{1+z^5}$ around the closed contour $P_{\rho,R} = C_R - L_{\rho,R} - C_\rho + [\rho, R]$ where $R > 1$, $0 < \rho, 1$ and

$$\begin{aligned}C_R &= \{Re^{i\theta} \mid 0 \le \theta \le 2\pi/5\},\\ L_{\rho,R} &= \{xe^{2\pi/5} \mid \rho \le x \le R\},\\ C_\rho &= \{\rho e^{i\theta} \mid 0 \le \theta \le 2\pi/5\},\\ [\rho, R] &= \{x \mid \rho \le x \le R\}.\end{aligned}$$

Inside this contour there is only one pole of $f(z)$, which is $z = e^{i\pi/5}$. By the residue theorem we have

$$\begin{aligned}\int_{P_{\rho,R}} f(z)dz &= 2\pi i \operatorname*{Res}_{z=e^{i\pi/5}} f(z)\\ &= 2\pi i \left(\frac{z^{1/5}}{5z^4}\right)_{z=e^{i\pi/5}}\\ &= \frac{2}{5}\pi i \left(z^{-19/5}|_{z=e^{i\pi/5}}\right)\\ &= \frac{2}{5}\pi i \left(e^{-\frac{19}{25}\pi i}\right)\\ &= -\frac{2}{5}\pi i e^{-\frac{6}{25}\pi i}\\ &= -\frac{2}{5}\pi i\left(\cos\frac{6}{25}\pi + i\sin\frac{6}{25}\pi\right)\\ &= \frac{2}{5}\sin\frac{6}{25}\pi - i\frac{2\pi}{5}\cos\frac{6}{25}\pi.\end{aligned}$$

On C_R we have:

$$\left|\int_{C_R} f(z)dz\right| \le \frac{R^{1/5}}{R^5 - 1}\frac{2\pi R}{5} \to 0 \text{ as } R \to \infty.$$

On C_ρ we have:

$$\left|\int_{C_\rho} f(z)dz\right| \leq \frac{\rho^{1/5}}{1-\rho^5}\frac{2\pi\rho}{5} \to 0 \text{ as } \rho \to 0.$$

Moreover on $L_{\rho,R}$ we have $z = xe^{2\pi i/5}$, $f(z)dz = f(x)e^{12\pi i/25}dx$ and hence

$$\int_{L_{\rho,R}} f(z)dz = e^{12\pi i/25}\int_\rho^R f(x)dx \to e^{12\pi i/5}\int_0^\infty \frac{x^{1/5}}{1+x^5}dx \text{ as } R \to \infty,\ \rho \to 0.$$

and

$$\int_{P_{\rho,R}} f(z)dz \to \left(1 - e^{12\pi i/25}\right)\int_0^\infty \frac{x^{1/5}}{1+x^5}dx \text{ as } R \to \infty,\ \rho \to 0.$$

Combining this with the residue calculation above, we find

$$\left[(1-\cos\frac{12\pi i}{25}) - i\sin\frac{12\pi i}{25}\right]\int_0^\infty \frac{x^{1/5}}{1+x^5}dx = \frac{2}{5}\sin\frac{6}{25}\pi - i\frac{2\pi}{5}\cos\frac{6}{25}\pi$$

which gives

$$\int_0^\infty \frac{x^{1/5}}{1+x^5}dx = \frac{2\pi}{5}\frac{\cos\frac{6\pi}{25}}{\sin\frac{12\pi}{25}} = \frac{\pi}{5}\frac{1}{\sin\frac{6\pi}{25}} = 0.91786...$$

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4) Consider the linear fractional transformation $f(z) = \dfrac{z-1}{z+1}$, and describe the images of the following sets under f.
i) The upper half plane.
ii) The unit circle.
iii) The x-axis.
iv) The y-axis.

Solution:
First write $f(z)$ in terms of x and y:

$$f(z) = \frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2} = u + iv.$$

i) When $y \geq 0$, we have $v \geq 0$. Hence the upper half plane maps onto the upper half plane. Here we used the fact that a nonconstant linear fractional transformation is onto.

ii) When $x^2 + y^2 = 1$, we have $u = 0$. Hence the unit circle maps onto the v-axis.

iii) Note that $f(-1) = \infty$, $f(0) = 1$ and $f(1) = 0$. The x-axis, which is a circle, must map onto the circle which passes through the points ∞, -1 and 1. Hence the image is the u-axis.

iv) Note again that $f(0) = -1$, $f(i) = i$ and $f(\infty) = 1$. The y-axis, which is a circle, must map onto the circle which passes through the points -1, i and 1. Hence the image is the unit circle.
