Date: 22 March 2003, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 206 Complex Calculus – Midterm Exam I
Solutions

1 Calculate the principal value of $\quad |\,(1+i)^{2+4i}\,| - \dfrac{2}{e^{\pi}}$.

Solution:

$$\begin{aligned}
(1+i)^{2+4i} &= \left[\sqrt{2}e^{i(\pi/4+2n\pi)}\right]^{2+4i} \\
&= \exp\left((2+4i)(\ln\sqrt{2}+i(\frac{\pi}{4}+2n\pi))\right) \\
&= \exp\left([2\ln\sqrt{2}-\pi-8n\pi]+i[4\ln\sqrt{2}+\frac{\pi}{2}+4n\pi]\right) \\
&= \frac{2}{e^{\pi(8n+1)}}\, e^{i(4\ln\sqrt{2}+\pi/2+4n\pi)}, \qquad n\in\mathbb{Z}.
\end{aligned}$$

The principal value corresponds to $n=0$, and the principal value of the absolute value of the above number is now seen to be $\dfrac{2}{e^{\pi}}$. Hence

$$|\,(1+i)^{2+4i}\,| - \frac{2}{e^{\pi}} = 0.$$

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2 Find all values of z for which we have $\cosh z = \dfrac{\sqrt{3}}{2}$. Use any method you like.

Solution: Method-1: $\cosh z = \cosh x\cos y + i\sinh x\sin y = \frac{\sqrt{3}}{2} + i0$.
So $\sinh x\sin y = 0$. If $\sin y = 0$, then $\cos y = \pm 1$ and from
$\cosh x\cos y = \frac{\sqrt{3}}{2}$, it follows that $\cosh x = \pm\frac{\sqrt{3}}{2}$, which is a contradiction since $\cosh x \geq 1$ for all $x\in\mathbb{R}$.
So we must have $\sinh x = 0$, which gives $x=0$. Then $\cosh x = 1$ and we end up with $\cos y = \frac{\sqrt{3}}{2}$.
From here it follows that $y = \pm\frac{\pi}{6} + 2n\pi$, where n is an integer.
Hence $z = \pm i(\dfrac{\pi}{6} + 2n\pi)$, where $n\in\mathbb{Z}$.

Method-2: Use $\cosh^{-1} z = \log[z + (z^2-1)^{1/2}]$ where you put $z = \frac{\sqrt{3}}{2}$. You get $\cosh^{-1}\dfrac{\sqrt{3}}{2} = \log[\dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}] = \log[\exp(i(\pm\dfrac{\pi}{6}+2n\pi))] = i(\pm\dfrac{\pi}{6}+2n\pi)$, which agrees with the previous answer when you note that n is any integer.

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3) Evaluate the integral $\displaystyle\int_{|z|=\frac{3}{2}} \frac{\cos \pi z}{z(z-1)^2(z-2)} dz.$

Solution:
Let $f(z) = \dfrac{\cos \pi z}{(z-1)^2(z-2)}$ and $g(z) = \dfrac{\cos \pi z}{z(z-2)}$.
Let C_1 be the positively oriented boundary of the region
$R_1 = \{z \in \mathbb{C} |\ \text{Re } z \geq 1/2, \text{and } |z| \leq 3/2\ \}$, and let C_2 be the positively oriented boundary of the region
$R_2 = \{z \in \mathbb{C} |\ \text{Re } z \leq 1/2, \text{and } |z| \leq 3/2\ \}$. Then

$$\begin{aligned}
\int_{|z|=\frac{3}{2}} \frac{\cos \pi z}{z(z-1)^2(z-2)} dz &= \int_{C_2} \frac{f(z)}{z} dz + \int_{C_1} \frac{g(z)}{(z-1)^2} dz \\
&= 2\pi i f(0) + 2\pi i g'(1) \\
&= 2\pi i(-\frac{1}{2}) + 2\pi i(0) \\
&= -\pi i.
\end{aligned}$$

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4) For any real constant $a \in \mathbb{R}$, calculate the integral $I_a = \displaystyle\int_0^{\pi} \cosh(a \cos \theta) \cos(a \sin \theta) d\theta.$

Solution:
From the Cauchy Integral Formula we find that $\displaystyle\int_{|z|=1} \frac{\cosh az}{z} dz = 2\pi i$. Now we evaluate this integral using the definition of complex integrals.
$z = e^{i\theta} = \cos \theta + i \sin \theta,\ -\pi \leq \theta \leq \pi.$
$dz = ie^{i\theta} d\theta$. Now let
$f(\theta) = \cosh(a \cos \theta) \cos(a \sin \theta)$, and $g(\theta) = \sinh(a \cos \theta) \sin(a \sin \theta)$. Note that f is even and g is odd, i.e. $f(-z) = f(z)$ and $g(-z) = -g(z)$. By direct calculation we find that

$$\begin{aligned}
\frac{\cosh az}{z} dz &= [\cosh(a \cos \theta + ia \sin \theta)] i d\theta \\
&= [\cosh(a \cos \theta) \cosh(ia \sin \theta) + \sinh(a \cos \theta) \sinh(ia \sin \theta)] i d\theta \\
&= [\cosh(a \cos \theta) \cos(a \sin \theta) + i \sinh(a \cos \theta) \sin(a \sin \theta)] i d\theta \\
&= [f(\theta) + ig(\theta)] i d\theta \\
&= -g(\theta) d\theta + if(\theta) d\theta
\end{aligned}$$

where we used the formulas $\cosh(iy) = \cos y$ and $\sinh(iy) = i \sin y$.

It now follows that

$$2\pi i = -\int_{-\pi}^{\pi} g(\theta) d\theta + i \int_{-\pi}^{\pi} f(\theta) d\theta = i \int_{-\pi}^{\pi} f(\theta) d\theta = 2i \int_0^{\pi} f(\theta) d\theta = 2iI_a.$$

Hence the answer is

$$I_a = \pi.$$
