Date: 3 May, 2003, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 206 Complex Calculus – Midterm Exam II
Solutions

1 Solve the following differential equation using Laplace transform techniques:

$$f''(t) - 4f'(t) + 3f(t) = 2\delta(t-4)$$

where $f(0) = 1$, $f'(0) = 4$. Here δ is the Dirac delta function, also known as the impulse function.

Solution: We apply Laplace transform to both sides of the differential equation using the formulas

$$\begin{aligned}
\mathfrak{L}(f(t)) &= F(s), \\
\mathfrak{L}(f'(t)) &= sF(s) - f(0) \\
&= sF(s) - 1, \\
\mathfrak{L}(f''(t)) &= s^2F(s) - sf(0) - f'(0) \\
&= s^2F(s) - s - 4, \\
\mathfrak{L}(\delta(t-4)) &= e^{-4s}.
\end{aligned}$$

The equation then becomes

$$(s^2 - 4s + 3)F(s) - s = 2e^{-4s}.$$

Noting that $s^2 - 4s + 3 = (s-1)(s-3)$, we find that

$$\begin{aligned}
\frac{1}{(s-1)(s-3)} &= \frac{1/2}{s-3} - \frac{1/2}{s-1}, \quad \text{and} \\
\frac{s}{(s-1)(s-3)} &= \frac{3/2}{s-3} - \frac{1/2}{s-1}.
\end{aligned}$$

Now solving for $F(s)$ we find that

$$\begin{aligned}
F(s) &= \frac{2e^{-4s}}{(s-1)(s-3)} + \frac{s}{(s-1)(s-3)} \\
&= e^{-4s}\left(\frac{1}{s-3} - \frac{1}{s-1}\right) + \frac{3/2}{s-3} - \frac{1/2}{s-1}.
\end{aligned}$$

We now recall the formula

$$\mathfrak{L}(H(t-\alpha)g(t-\alpha)) = e^{-\alpha s}\mathfrak{L}(g(t))$$

where H is the Heaviside function and g is any function. Using this we easily take Laplace inverse transform of both sides of the equation for $F(s)$ and get

$$f(t) = H(t-4)\left(e^{3(t-4)} - e^{t-4}\right) + \frac{3}{2}e^{3t} - \frac{1}{2}e^{t}.$$

Check that this is actually the solution of the given differential equation. For this you may need to know that $H(t)' = \delta(t)$ and that for any differentiable function $h(t)$, you have

$$h(t)\delta'(t-\alpha) = h(\alpha)\delta'(t-\alpha) - h'(\alpha)\delta(t-\alpha).$$

This last property follows from calculating the derivative of $h(t)\delta(t-\alpha)$ in two different ways as follows. First using the product rule for differentiation you have

$$\begin{aligned}(h(t)\delta(t-\alpha))' &= h'(t)\delta(t-\alpha) + h(t)\delta'(t-\alpha)\\ &= h'(\alpha)\delta(t-\alpha) + h(t)\delta'(t-\alpha).\end{aligned}$$

On the other hand you have

$$\begin{aligned}(h(t)\delta(t-\alpha))' &= (h(\alpha)\delta(t-\alpha))'\\ &= h(\alpha)\delta'(t-\alpha)\end{aligned}$$

Putting these together you obtain the claimed property.

---

2 Solve the following difference equation:

$$f(n+2) - f(n) = n^2,$$

where $f(0) = 2$ and $f(1) = 0$.

Solution: We take the z-transform of both sides using the formulas

$$\begin{aligned}\mathcal{Z}(f(n)) &= F(z),\\ \mathcal{Z}(f(n+2)) &= z^2F(z) - z^2f(0) - zf(1)\\ &= z^2F(z) - 2z^2,\\ \mathcal{Z}(n^2) &= \frac{z(z+1)}{(z-1)^3}.\end{aligned}$$

The equation now becomes

$$z^2F(z) - 2z^2 - F(z) = \frac{z(z+1)}{(z-1)^3}.$$

Solving for $F(z)$ we find

$$\begin{aligned}F(z) &= \frac{z}{(z-1)^4} + \frac{2z^2}{(z-1)(z+1)}\\ &= \frac{z}{(z-1)^4} + \frac{z}{(z-1)} + \frac{z}{(z+1)}.\end{aligned}$$

Taking the inverse z-transform of both sides we get

$$\begin{aligned}f(n) &= \mathrm{Res}_{z=1}\frac{z^n}{(z-1)^4} + 1 + (-1)^n\\ &= \frac{n(n-1)(n-2)}{3!} + 1 + (-1)^n.\end{aligned}$$

---

3) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{1+x^9}dx.$

Solution:
We integrate the function $f(z) = z^2/(1+z^9)$ around the closed contour $P_R = C_R + L_R + [0,R]$ where $R > 1$ and

$$\begin{aligned}C_R &= \{Re^{i\theta} \mid 0 \le \theta \le 2\pi/9\},\\ -L_R &= \{xe^{2\pi/9} \mid 0 \le x \le R\},\\ [0,R] &= \{x \mid 0 \le x \le R\}.\end{aligned}$$

Inside this contour there is only one pole of $f(z)$, which is $z = e^{i\pi/9}$. By the residue theorem we have

$$\begin{aligned}\int_{P_R} f(z)dz &= 2\pi i \text{Res}_{z=e^{i\pi/9}} f(z)\\ &= 2\pi i\left(\frac{z^2}{9z^8}\right)_{z=e^{i\pi/9}}\\ &= 2\pi i\left(\frac{1}{9}e^{-i(2\pi/3)}\right)\\ &= (2\pi i)(-1/18 - i\sqrt{3}/18)\\ &= \frac{\pi\sqrt{3}}{9} - i\frac{\pi}{9}.\end{aligned}$$

Since $2 < 9 - 1$, the integral of $f(z)$ on C_R converges to zero as R goes to infinity. By direct calculation we see that

$$\int_{L_R} f(z)dz = -e^{i2\pi/3}\int_{[0,R]} f(z)dz.$$

Let

$$I = \int_0^\infty \frac{x^2}{1+x^9}dx.$$

Then after taking limits as R goes to infinity we get

$$\begin{aligned}\frac{\pi\sqrt{3}}{9} - i\frac{\pi}{9} &= (1 - e^{i2\pi/3})I\\ &= (3/2 - i\sqrt{3}/2)I.\end{aligned}$$

Hence

$$I = \frac{2\pi}{9\sqrt{3}} = \frac{2\sqrt{3}\pi}{27}.$$

---

4) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^{1/2}}{(1+x)^3}dx$.

Solution:
For this we will set $f(z) = z^{1/2}/(1+z)^3$ and use the usual closed path of figure 70 in your text book on page 224. First note that

$$\left|\int_{C_R} \frac{z^{1/2}}{(1+z)^3}dz\right| \leq 2\pi\frac{R^{\,3/2}}{(R-1)^3} \mapsto 0 \text{ as } R \mapsto \infty$$

and

$$\left|\int_{C_\rho} \frac{z^{1/2}}{(1+z)^3}dz\right| \leq 2\pi\frac{\rho^{\,3/2}}{(1-\rho)^3} \mapsto 0 \text{ as } \rho \mapsto 0.$$

Let $L_{R,\,\rho}$ denote the line along $z = xe^{2\pi i}$ as x ranges from R to ρ. We then have

$$\begin{aligned}\int_{L_{R,\,\rho}} f(z)dz &= -\int_{-L_{R,\,\rho}} f(z)dz\\ &= -\int_\rho^R \frac{z^{1/2}e^{2\pi i/2}}{(1+z)^3}dz\\ &= \int_\rho^R \frac{z^{1/2}}{(1+z)^3}dz.\end{aligned}$$

Let $P_{R,\ \rho} = [\rho, R] + C_R + C_\rho + L_{R,\ \rho}$. By residue formula we get

$$\begin{aligned}
\int_{P_{R,\ \rho}} \frac{z^{1/2}}{(1+z)^3} dz &= 2\pi i \mathrm{Res}_{z=-1} \frac{z^{1/2}}{(1+z)^3} \\
&= (2\pi i)\frac{1}{2!}\ \frac{d^2 z^{1/2}}{dz^2}\bigg|_{z=-1} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Taking limits of both sides as $R \mapsto \infty$, $\rho \mapsto 0$ and noting that the integral on $L_{R,\ \rho}$ has the same limit as the integral on $[\rho, R]$ we get

$$\int_0^\infty \frac{x^{1/2}}{(1+x)^3} dx = \frac{\pi}{8}$$
