Math 206 Complex Calculus
Quiz-2
Solutions

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1-a) Given $\cos^{-1} z = -i\log(z + i(1-z^2)^{1/2})$, calculate $\cos^{-1}(2i)$.

Solution 1-a:

$$\begin{aligned}
\cos^{-1}(2i) &= -i\log\left(2i + i(1-(2i)^2)^{1/2}\right) \\
&= -i\log\left(2i + i(1-(-4))^{1/2}\right) \\
&= -i\log\left(2i + i(5)^{1/2}\right) \\
&= -i\log\left(2i \pm \sqrt{5}i\right).
\end{aligned}$$

Taking the + sign, we have:

$$\begin{aligned}
\cos^{-1}(2i) &= -i\log\left(2i + \sqrt{5}i\right). \\
&= -i\log\left((2+\sqrt{5})i\right). \\
&= -i\log\left((2+\sqrt{5})\exp(i(\pi/2 + 2n\pi))\right) \\
&= -i\left(\ln(2+\sqrt{5}) + i(\pi/2 + 2n\pi)\right) \\
&= (1/2 + 2n)\pi - i\ln(2+\sqrt{5}), \quad n \in \mathbb{Z}.
\end{aligned}$$

Taking the − sign, we have:

$$\begin{aligned}
\cos^{-1}(2i) &= -i\log\left(2i - \sqrt{5}i\right). \\
&= -i\log\left((\sqrt{5}-2)(-i)\right). \\
&= -i\log\left((\sqrt{5}-2)\exp(i(-\pi/2 + 2n\pi))\right) \\
&= -i\left(\ln(\sqrt{5}-2) + i(-\pi/2 + 2n\pi)\right) \\
&= (-1/2 + 2n)\pi - i\ln(\sqrt{5}-2) \\
&= (-1/2 + 2n)\pi + i\ln(2+\sqrt{5}), \quad n \in \mathbb{Z}.
\end{aligned}$$

Hence we have

$$\cos^{-1}(2i) \;=\; (\pm 1/2 + 2n)\pi \mp i\ln(2+\sqrt{5}), \quad n \in \mathbb{Z}.$$

1-b) Given $\sinh^{-1} z = \log(z + (z^2+1)^{1/2})$, calculate $\sinh^{-1}(2i)$.

Solution 1-b:

$$\begin{aligned}
\sinh^{-1}(2i) &= \log\left(2i + ((2i)^2+1)^{1/2}\right) \\
&= \log\left(2i + ((-4)+1)^{1/2}\right) \\
&= \log\left(2i + (-3)^{1/2}\right) \\
&= \log\left(2i \pm \sqrt{3}i\right) \\
&= \log\left((2 \pm \sqrt{3})i\right).
\end{aligned}$$

Taking the + sign, we have:

$$\begin{aligned}
\sinh^{-1}(2i) &= \log\left((2+\sqrt{3})i\right) \\
&= \log\left((2+\sqrt{3})\exp(i(\pi/2+2n\pi))\right) \\
&= \ln(2+\sqrt{3}) + i(\pi/2+2n\pi) \\
&= \ln(2+\sqrt{3}) + i(1/2+2n)\pi, \quad n \in \mathbb{Z}.
\end{aligned}$$

Taking the − sign, we have:

$$\begin{aligned}
\sinh^{-1}(2i) &= \log\left((2-\sqrt{3})i\right) \\
&= \log\left((2-\sqrt{3})\exp(i(\pi/2+2n\pi))\right) \\
&= \ln(2-\sqrt{3}) + i(\pi/2+2n\pi) \\
&= -\ln(2+\sqrt{3}) + i(1/2+2n)\pi, \quad n \in \mathbb{Z}.
\end{aligned}$$

Hence we have

$$\sinh^{-1}(2i) \;=\; \pm\ln(2+\sqrt{3}) + i(1/2+2n)\pi, \quad n \in \mathbb{Z}.$$