Math 206 Complex Calculus
Quiz-5
Solutions

May 15, 2003 Ali Sinan Sertöz

1-a) Let $D = \{(x,y) \in \mathbb{R}^2 |\ x, y \geq 0\ \}$. Find a bounded harmonic function $T(x,y)$ on D such that $T(x,0) = 0$ and $T(0,y) = 1$.

Solution 1-a: Map the region D by $\log z = \ln r + i\theta = u + iv$ onto the region $E = \{(u,v) \in \mathbb{R}^2 |\ 0 \leq v \leq \pi/2\ \}$. Consider the function $H(u,v) = 2v/\pi$ on E. This is harmonic on E and satisfies the given boundary conditions. Let $T(x,y) = H(u(x,y), v(x,y))$. Putting in $v = \theta = \arctan(y/x)$ we get $T(x,y) = (2/\pi)\arctan(y/x)$ as the required bounded harmonic function.
Note: This is a simplified version of Exercise 4 on page 307.

1-b) Let $D = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1,\ x, y \geq 0\ \}$. Find a harmonic function $T(x,y)$ on D such that $T(x,0) = 0$, $T(0,y) = 1$ and $\partial T/\partial \vec{n} = 0$ along the circular part of the boundary.

Solution 1-b: The function $\log z = \ln r + i\theta = u + iv$ maps D onto the region $E = \{(u,v) \in \mathbb{R}^2 |\ u \leq 0,\ 0 \leq v \leq \pi/2$. The function $H(u,v) = 2v/\pi$ is harmonic on E and satisfies the given boundary conditions. Define $T(x,y) = H(u(x,y), v(x,y))$. Putting in $v = \theta = \arctan(y/x)$ we get $T(x,y) = (2/\pi)\arctan(y/x)$ as the required bounded harmonic function.
Note: This is a simplified version of Exercise 5 on page 308.
