Page 71 Exercise 2: Show that $e^{iz} = \cos z + i \sin z$ for every complex number $z$.
Solution: We can do this easily using equation (1) on page 69

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \frac{e^{iz} - e^{-iz}}{2i} = e^{iz}.$$ 

We can also use equations (11) and (12) on page 70 to write the real and imaginary parts of $\cos z + i \sin z$ and after simplifying it obtain the real and imaginary parts of $e^{iz}$ where we use equation (3) on page 66:

$$\cos z + i \sin z = \cos x \cosh y - i \sin x \sinh y + i (\sin x \cosh y + i \cos x \sinh y)$$
$$= (\cos x + i \sin x)(\cosh y - \sinh y)$$
$$= e^{ix} e^{-y}$$
$$= e^{iz}.$$

Page 72 Exercise 11: Show that neither $\sin \pi$ nor $\cos \pi$ is an analytic function of $z$ anywhere.
Solution: When $z = x + iy$, $\sin z = \sin x \cosh y + i \cos x \sinh y$. Putting $\pi = x - iy$ for $z$ we obtain $\sin \pi = \sin x \cosh y - i \cos x \sinh y$. We check that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold only when $z = (2n + 1/2)\pi$, for $n \in Z$. These are isolated points. A function is called analytic when Cauchy-Riemann equations hold in an open set. See section 20 on page 55. So $\sin \pi$ is not analytic anywhere.

Similarly $\cos \pi = \cos x \cosh y + i \sin x \sinh y = u + iv$, and the Cauchy-Riemann equations hold when $z = n\pi$ for $n \in Z$. Thus $\cos \pi$ is not analytic anywhere, for the same reason as above.

Page 80 Exercise 13: Show that
(a) the function $\log(z - i)$ is analytic everywhere except on the half line $y = 1$, $x \leq 0$.
(b) the function $\frac{\log(z + 4)}{z^2 + i}$ is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.
Solution: (a) $\log w$ is analytic for every value of $w = u + iv$ except on the half line $v = 0, u \leq 0$. Putting $z - i = x + i(y - 1) = u + iv$, we see that $\log(z - i)$ is analytic everywhere except on the half line $y - 1 = 0$, $x \leq 0$.

(b) As in part (a), $\log(z + 4)$ is analytic everywhere except on the half line $y = 0$, $x \leq -4$. We should also exclude the points where the denominator vanishes. For this we solve for
\[ z^2 = -i = \exp(i(-\pi/2 + 2n\pi)) \]. This gives \( z = \pm(1 - i)/\sqrt{2} \). See section 7, on page 19, for finding such roots.

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**Page 85 Exercise 11:** Solve the equation \( \sin z = 2 \) for \( z \)

(a) by equating real and imaginary parts in that equation.

(b) using expression for \( \sin^{-1} z \).

**Solution:**

(a) Set \( \sin z = \sin x \cosh y + i \cos x \sinh y = 2 \). This gives

\[
\sin x \cosh y = 2,
\cos x \sinh y = 0.
\]

The second equation holds when \( x = (n + 1/2)\pi \) or when \( y = 0 \). But when \( y = 0 \), the first equation becomes \( x = 2 \), which has no solution. So we must have \( x = (n + 1/2)\pi \). In that case the first equation becomes \( (-1)^n \cosh y = 2 \). But \( \cosh y \) is always positive, so \( n \) must be an even integer. We then solve for \( \cosh y = e^{y+e^{-y}} = 2 \). Putting \( w = e^y \) in this equation and solving for the resulting quadratic equation, we get \( w = 2 \pm \sqrt{3} \). Then \( y = \pm \ln(2 + \sqrt{3}) \). Here we use the observation that \( 2 - \sqrt{3} = 1/(2 + \sqrt{3}) \). Hence the solution set is \( z = (2n + 1/2)\pi \pm i \ln(2 + \sqrt{3}) \).

(b) Putting \( z = 2 \) into the formula \( \sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}] \) we get

\[
\sin^{-1} 2 = -i \log(2i \pm i\sqrt{3})
= -i \log(i(2 \pm \sqrt{3}))
= -i \log[(2 \pm \sqrt{3})e^{i(\pi/2 + 2n\pi)}]
= -i[\ln(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi)]
= (\pi/2 + 2n\pi) \pm i \ln(2 + \sqrt{3}).
\]

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**Page 85 Exercise 12:** Solve the equation \( \cos z = \sqrt{2} \).

**Solution:** The formula for inverse cosine is \( \cos^{-1} z = -i \log[z + i(1 - z^2)^{1/2}] \). Putting \( z = \sqrt{2} \), we get

\[
\cos^{-1} \sqrt{2} = -i \log[\sqrt{2} \pm 1]
= \pm i \log[\sqrt{2} + 1]
= \pm i \log[(\sqrt{2} + 1)e^{i2n\pi}]
= \pm i[\ln(\sqrt{2} + 1) + i2n\pi]
= 2n\pi \pm i \ln(\sqrt{2} + 1).
\]