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MATH 206
HOMEWORK 4 SOLUTIONS

Page 71 Exercise 2: Show that $e^{iz} = \cos z + i \sin z$ for every complex number z.
Solution: We can do this easily using equation (1) on page 69

$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = e^{iz}.$$

We can also use equations (11) and (12) on page 70 to write the real and imaginary parts of $\cos z + i \sin z$ and after simplifying it obtain the real and imaginary parts of e^{iz} where we use equation (3) on page 66:

$$\begin{aligned} \cos z + i \sin z &= \cos x \cosh y - i \sin x \sinh y + i(\sin x \cosh y + i \cos x \sinh y) \\ &= (\cos x + i \sin x)(\cosh y - \sinh y) \\ &= e^{ix} e^{-y} \\ &= e^{iz}. \end{aligned}$$

Page 72 Exercise 11: Show that neither $\sin \overline{z}$ nor $\cos \overline{z}$ is an analytic function of z anywhere.
Solution: When $z = x + iy$, $\sin z = \sin x \cosh y + i \cos x \sinh y$. Putting $\overline{z} = x - iy$ for z we obtain $\sin \overline{z} = \sin x \cosh y - i \cos x \sinh y = u + iv$. We check that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold only when $z = (2n + 1/2)\pi$, for $n \in \mathbb{Z}$. These are isolated points. A function is called analytic when Cauchy-Riemann equations hold in an open set. See section 20 on page 55. So $\sin \overline{z}$ is not analytic anywhere.

Similarly $\cos \overline{z} = \cos x \cosh y + i \sin x \sinh y = u + iv$, and the Cauchy-Riemann equations hold when $z = n\pi$ for $n \in \mathbb{Z}$. Thus $\cos \overline{z}$ is not analytic anywhere, for the same reason as above.

Page 80 Exercise 13: Show that
(a) the function $Log(z - i)$ is analytic everywhere except on the half line $y = 1$, $x \leq 0$.
(b) the function $\dfrac{Log(z + 4)}{z^2 + i}$ is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.
Solution: (a) $Log\, w$ is analytic for every value of $w = u + iv$ except on the half line $v = 0$, $u \leq 0$. Putting $z - i = x + i(y - 1) = u + iv$, we see that $Log(z - i)$ is analytic everywhere except on the half line $y - 1 = 0$, $x \leq 0$.

(b) As in part (a), $Log(z + 4)$ is analytic everywhere except on the half line $y = 0$, $x \leq -4$. We should also exclude the points where the denominator vanishes. For this we solve for

$z^2 = -i = exp(i(-\pi/2 + 2n\pi))$. This gives $z = \pm(1-i)/\sqrt{2}$. See section 7, on page 19, for finding such roots.

Page 85 Exercise 11: Solve the equation $\sin z = 2$ for z
(a) by equating real and imaginary parts in that equation.
(b) using expression for $\sin^{-1} z$.
Solution: (a) Set $\sin z = \sin x \cosh y + i \cos x \sinh y = 2$. This gives
$\sin x \cosh y = 2$,
$\cos x \sinh y = 0$.
The second equation holds when $x = (n + 1/2)\pi$ or when $y = 0$. But when $y = 0$, the first equation becomes $\sin x = 2$, which has no solution. So we must have $x = (n + 1/2)\pi$. In that case the first equation becomes $(-1)^n \cosh y = 2$. But $\cosh y$ is always positive, so n must be an even integer. We then solve solve for $\cosh y = \frac{e^y + e^{-y}}{2} = 2$. Putting $w = e^y$ in this equation and solving for the resulting quadratic equation, we get $w = 2 \pm \sqrt{3}$. Then $y = \pm \ln(2+\sqrt{3})$. Here we use the observation that $2-\sqrt{3} = 1/(2+\sqrt{3})$. Hence the solution set is $z = (2n + 1/2)\pi \pm i \ln(2 + \sqrt{3})$.

(b) Putting $z = 2$ into the formula $\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]$ we get

$$\begin{aligned}
\sin^{-1} 2 &= -i \log(2i \pm i\sqrt{3}) \\
&= -i \log(i(2 \pm \sqrt{3})) \\
&= -i \log[(2 \pm \sqrt{3})e^{i(\pi/2+2n\pi)}] \\
&= -i[\ln(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi)] \\
&= (\pi/2 + 2n\pi) \pm i \ln(2 + \sqrt{3}).
\end{aligned}$$

Page 85 Exercise 12: Solve the equation $\cos z = \sqrt{2}$.
Solution: The formula for inverse cosine is $\cos^{-1} z = -i \log[z + i(1 - z^2)^{1/2}]$. Putting $z = \sqrt{2}$, we get

$$\begin{aligned}
\cos^{-1} \sqrt{2} &= -i \log[\sqrt{2} \pm 1] \\
&= \pm i \log[\sqrt{2} + 1] \\
&= \pm i \log[(\sqrt{2} + 1)e^{i2n\pi}] \\
&= \pm i[\ln(\sqrt{2} + 1) + i2n\pi] \\
&= 2n\pi \pm i \ln(\sqrt{2} + 1).
\end{aligned}$$
