MATH 206, HW#3
SOLUTIONS

p.42: 4. If $\lim_{z\to 0}(\frac{z}{\bar{z}})^2$ exists, then

$$\lim_{x\to 0}(\frac{x+i0}{x-i0})^2 = \lim_{(x,x)\to(0,0)}(\frac{x+ix}{x-ix})^2$$

should hold. (In fact, one should have the same limit no matter how z approaches the origin.) However,

$$\lim_{x\to 0}(\frac{x+i0}{x-i0})^2 = \lim_{x\to 0}(\frac{x}{x})^2 = 1$$

whereas

$$\lim_{(x,x)\to(0,0)}(\frac{x+ix}{x-ix})^2 = \lim_{x\to 0}\frac{(x+ix)^2}{(x-ix)^2} = \lim_{x\to 0}\frac{2xi}{-2xi} = -1.$$

p.47: 1.

a. Since $f(z)$ is a polynomial, $f'(z) = 6z - 2$

b. By chain rule, $f'(z) = 3(1-4z^2)^2(-8z) = -24z(1-4z^2)^2$

c. By ratio rule, $f'(z) = \frac{1(2z+1)-2(z-1)}{(2z+1)^2} = \frac{3}{(2z+1)^2}$

c. By ratio and chain rule, $f'(z) = \frac{4(1+z^2)^3(2z)z^2 - 2z(1+z^2)^4}{z^4} = (1+z^2)^3\ \frac{2(3z^2-1)}{z^3}$.

p.55: 9.

a. Let $z_0 = r_0 e^{i\theta_0}$. Using the Cauchy-Riemann equations in polar form

$$u_r = \frac{1}{r}v_\theta,\ v_r = -\frac{1}{r}u_\theta$$

in $f'(z_0) = e^{-i\theta_0}[u_r(r_0,\theta_0) + iv_r(r_0,\theta_0)]$, we obtain

$$f'(z_0) = e^{-i\theta_0}[\frac{1}{r_0}v_\theta(r_0,\theta_0) - i\frac{1}{r_0}u_\theta(r_0,\theta_0)] = \frac{-i}{z_0}[u_\theta(r_0,\theta_0) + iv_\theta(r_0,\theta_0)],$$

where we put z_0 for $r_0 e^{i\theta_0}$.

b. We have $f(z) = 1/z$ so that $u(r,\theta) = \frac{1}{r}\cos(\theta)$ and $v(r,\theta) = -\frac{1}{r}\sin(\theta)$. This give

$$u_\theta(r,\theta) = -\frac{1}{r}\sin(\theta),\ v_\theta(r,\theta) = -\frac{1}{r}\cos(\theta)$$

so that

$$f'(z) = \frac{-i}{z}[-\frac{1}{r}\sin(\theta) - i\frac{1}{r}\cos(\theta)] = -\frac{1}{z}\frac{1}{r}[\cos(\theta) - i\sin(\theta)] = -\frac{1}{z^2}.$$

p.62: 4. The functions $f(z)$ in all parts of this question are rational functions of z. A point z_0 is a singular point of a rational function if and only if it is a zero of the denominator polynomial *after all cancellations with the numerator polynomial is carried out.*

a. Singular points are $0, \pm i$ since these are the three zeros of the denominator polynomial $z(z^2+1)$ and since neither of them is a zero of $z + 1/2$.

b. Singular points are 1, 2 since these are the three zeros of the denominator polynomial $z^2 - 3z + 2$ and since neither of them is a zero of $z^3 + i$, which are $\frac{1}{2}(\sqrt{3}+i),\ \frac{1}{2}(-\sqrt{3}+i), -i$.

c. Singular points are $-2,\ -1 \pm i$ since these are the three zeros of the denominator polynomial $(z+2)(z^2+2z+2)$ and since neither of them is a zero of z^2+1, which are $\pm i$.

p.64: 18. We have

$$f(x+iy) = \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1)^2+y^2} = \frac{x^2-1+y^2}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2}.$$

The level curves are

$$\frac{x^2-1+y^2}{(x+1)^2+y^2} = c_1, \quad \frac{2y}{(x+1)^2+y^2} = c_2$$

for arbitrary real numbers c_1, c_2. Writing

$$x^2 - 1 + y^2 = c_1[(x+1)^2+y^2], \quad 2y = c_2[(x+1)^2+y^2]$$

we obtain, for $c_1 \neq 1$ and $c_2 \neq 0$, the equalities

$$(x - \frac{c_1}{1-c_1})^2 + y^2 = \frac{1}{(c_1-1)^2}, \quad (x+1)^2 + (y - \frac{1}{c_2})^2 = \frac{1}{c_2^2}.$$

The first equation gives circles with center at $(\frac{c_1}{1-c_1}, 0)$ and with radius $\frac{1}{|1-c_1|}$. These circles are tangent to the line $x = -1$ at $(-1, 0)$.

The second equation gives circles with center at $(-1, \frac{1}{c_2})$ with radius $\frac{1}{|c_2|}$. These circles are tangent to the x-axis at $(-1, 0)$ and are orthogonal to the family of circles given by the first equation.

When $c_1 = 1$, the corresponding level curve is the line $x = -1$ and when $c_2 = 0$, it is the line $y = 0$.