MATH 206
HOMEWORK-7 SOLUTIONS

p. 188: 1. a) The function $\dfrac{1}{z+z^2}$ is analytic in $0 < |z| < 1$ so that it has a Laurent series expansion at $z = 0$:

$$\frac{1}{z+z^2} = \frac{1}{z(z+1)} = \left(\frac{1}{z}\right)\left(\frac{1}{1-(-z)}\right) = \frac{1}{z}(1 - z + z^2 - ...).$$

The residue is hence $b_1 = 1$.

b) The function $zcos(\dfrac{1}{z})$ is analytic in $0 < |z| < \infty$ so that it has a Laurent series expansion at $z = 0$:

$$zcos(\frac{1}{z}) = z(1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - ...).$$

Hence, $b_1 = -1/2$.

c) The function $\dfrac{z - sin\, z}{z}$ is analytic in $0 < |z| < \infty$ so that it has a Laurent series expansion at $z = 0$:

$$\frac{z - sin\, z}{z} = 1 - \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...) = \frac{z^2}{3!} - \frac{z^4}{5!} - ...$$

Hence $b_1 = 0$.

d) The function $\dfrac{cot\, z}{z^4} = \dfrac{cos\, z}{z^4 sin\, z}$ is analytic in $0 < |z| < \pi$ so that it has a Laurent series expansion at $z = 0$. The coefficient of z^3 in the Laurent series expansion of $\frac{cos\, z}{sin\, z}$ will give the residue b_1. By long division

$$\frac{cos\, z}{sin\, z} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...} = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - ...$$

and $b_1 = -1/45$.

e) The function $\frac{sinh\, z}{z^4(1-z^2)}$ is analytic in $0 < |z| < 1$ so that it has a Laurent series expansion at $z = 0$:

$$\frac{sinh\, z}{z^4(1-z^2)} = \frac{1}{z^4}(z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...)(1 + z^2 + z^4 + ...) = \frac{1}{z^4}[z + (1 + \frac{1}{3!})z^3 + (1 + \frac{1}{3!} + \frac{1}{5!})z^5 + ...].$$

Hence $b_1 = 1 + \frac{1}{3!} = \frac{7}{6}$.

p. 208: 7.

$$P.V. \int_{-\infty}^{\infty} \frac{x\, dx}{(x^2+1)(x^2+2x+2)} = ?$$

Let

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)}$$

which has singularities $i, -1+i$ in the positively oriented contour consisting of the real segment $[-R, R]$ and the semicircle C_R of radius R in the upper half of the complex plane, with $R > \sqrt{2}$. By the Residue Theorem

$$P.V. \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$$

$$= Re[2\pi\, i(B_1+B_2)] - Re(\lim_{R\to\infty} \int_{C_R} f(z)dz),$$

where B_1 and B_2 are the residues at i and $-1+i$ of $f(z)$, respectively. Now

$$\begin{aligned} B_1 &= \frac{z}{(z+i)(z^2+2z+2)}|_{z=i} = \frac{i}{2i(1+2i)} = \frac{1-2i}{10}, \\ B_2 &= \frac{z}{(z^2+1)(z+1+i)}|_{z=-1+i} = \frac{-1+3i}{10}. \end{aligned}$$

On C_R, $z = R\,exp(i\theta),\ 0 \le \theta \le \pi$ so that

$$|\int_{C_R} f(z)dz| \le \frac{\pi R^2}{(R^2-1)[(R-1)^2-1]}.$$

It follows that

$$Re(\lim_{R\to\infty} \int_{C_R} f(z)dz) = 0$$

and

$$P.V. \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = Re[2i\pi(\frac{1-2i}{10} + \frac{-1+3i}{10})] = -\frac{\pi}{5}.$$

p. 215: 12. The function $f(z) = exp(iz^2)$ is analytic everywhere. By Cauchy-Goursat Theorem its contour integral along the positively oriented contour consisting of the real segment $C_1 : z = x, 0 \le x \le R$, the segment of a circle $C_R : z = Rexp(i\theta), 0 \le \theta \le \pi/4$, and the ray C_2 where $-C_2$ is parameterized as: $z = rexp(i\pi/4), 0 \le r \le R$, has value zero. Thus,

$$\begin{aligned} 0 &= \int_0^R exp(ix^2)\,dx + \int_R^0 exp(ir^2 e^{i\pi/2})\,dr + \int_{C_R} exp(iz^2)\,dz \\ &= \int_0^R exp(ix^2)\,dx - \frac{1+i}{\sqrt{2}} \int_0^R exp(-r^2)\,dr + \int_{C_R} exp(iz^2)\,dz, \end{aligned}$$

where the first two integrals are integrals of $exp(iz^2)$ along C_1 and C_2, respectively. Since $exp(ix^2) = cos\,x^2 + i\,sin\,x^2$, equating the real and imaginary parts, we have

$$\int_0^R cos\,x^2\,dx = \frac{1}{\sqrt{2}} \int_0^R exp(-r^2)\,dr + Re[\int_{C_R} exp(iz^2)\,dz] \quad (1)$$

$$\int_0^R sin\,x^2\,dx = \frac{1}{\sqrt{2}} \int_0^R exp(-r^2)\,dr + Im[\int_{C_R} exp(iz^2)\,dz]. \quad (2)$$

Now, on C_R, $z = Rexp(i\theta),\ 0 \le \theta \le \pi/4$ so that

$$|\int_{C_R} exp(iz^2)\,dz| \le \int_0^{\pi/4} |exp(iR^2\,e^{i2\theta})iRe^{i\theta}\,d\theta| \le R \int_0^{\pi/4} exp(-R^2\,sin(2\theta))\,d\theta$$

$$= \frac{R}{2}\int_0^{\pi/2} exp(-R^2\, sin(\phi))\, d\phi \le \frac{R}{2}\frac{\pi}{2R^2} = \frac{\pi}{4R},$$

where the last inequality is obtained by Jordan's inequality. Thus,

$$\lim_{R\to\infty} Re[\int_{C_R} exp(iz^2)\, dz] = 0,\ \lim_{R\to\infty} Im[\int_{C_R} exp(iz^2)\, dz] = 0$$

and we obtain, by taking limits as $R \to \infty$ in (1) and (2),

$$\int_0^\infty cos\, x^2\, dx = \int_0^\infty sin\, x^2\, dx = \frac{1}{\sqrt{2}}\int_0^\infty exp(-r^2)\, dr.$$

To evaluate the last integral, we consider

$$(\int_0^\infty exp(-x^2)\, dx)(\int_0^\infty exp(-y^2)\, dy) = \int_0^\infty\int_0^\infty exp[-(x^2+y^2)]\, dx\, dy.$$

Changing to polar coordinates $x = r\, cos\theta,\ y = r\, sin\theta$ (See a book on calculus!), where $0 \le r < \infty$ and $0 \le \theta \le \pi/2$ (i.e., the first quadrant), we have

$$\int_0^\infty\int_0^\infty exp[-(x^2+y^2)]\, dx\, dy = \int_0^{\pi/2}\int_0^\infty exp(-r^2)\, r\, dr\, d\theta = \pi/4.$$

This gives

$$\int_0^\infty exp(-x^2)\, dx = \frac{\sqrt{\pi}}{2},$$

and

$$\int_0^\infty cos\, x^2\, dx = \int_0^\infty sin\, x^2\, dx = \frac{1}{\sqrt{2}}\frac{\sqrt{\pi}}{2}.$$

p. 226: 2. To evaluate the integral

$$\int_0^\infty \frac{x^a\, dx}{(x^2+1)^2},\ -1 < a < 3,$$

we let

$$f(z) = \frac{exp(a log\, z)}{(z^2+1)^2},\ |z| > 0,\ -\pi/2 < arg\, z < 3\pi/2$$

and consider its integral along the positively oriented contour consisting of the segment $L_1 : z = r,\ \rho \le r \le R$, the large semi-circle $C_R : z = R\, exp(i\theta),\ 0 \le \theta \le \pi$, the segment L_2 where $-L_2$ is parameterized as: $z = -r,\ \rho \le r \le R$, and the small semi-circle C_ρ where $-C_\rho$ is parameterized as: $z = \rho\, exp(i\theta),\ 0 \le \theta \le \pi$. The function $f(z)$ has only one singularity inside the contour at $z = i$ so that, by the Residue Theorem,

$$\int_\rho^R \frac{r^a}{(r^2+1)^2}\, dr - \int_\rho^R \frac{exp(a\, ln\, r + ia\pi)}{(r^2+1)^2}\, dr = 2i\pi\phi'(i) - \int_{C_\rho} f(z)\, dz - \int_{C_R} f(z)\, dz,\ (3)$$

where

$$f(z) = \frac{\phi(z)}{(z-i)^2},\quad \phi(z) = \frac{exp(a\, log\, z)}{(z+i)^2}.$$

Now,

$$\phi'(z) = \frac{a\, exp(a\, log\, z)(z+i) - 2z\, exp(a\, log\, z)}{z(z+i)^3},\ \phi'(i) = \frac{exp(ia\pi/2)(2ia-2i)}{-i\, 8i}.$$

Moreover, on C_R and on C_ρ, it is easy to show that

$$|\int_{C_R} f(z)\,dz| \leq \frac{\pi\, R^{1+a-4}}{(1-\frac{1}{R^2})^2},\ |\int_{C_\rho} f(z)\,dz| \leq \frac{\pi\,\rho^{1+a}}{(1-\rho^2)^2},$$

so that by $1+a-4<0$ and by $1+a>0$, the limits as $R\to\infty$ and as $\rho\to 0$ are both zero. It now follows from (3) on taking limits that

$$[1+exp(ia\pi)]\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{2\pi i^2(a-1)\,exp(ia\pi/2)}{4}.$$

From this it follows that

$$\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{\pi\,(1-a)}{4cos(a\pi/2)}.$$

p. 227: 4. We let

$$f(z) = \frac{(log\,z)^2}{(z^2+1)},\ |z|>0,\ -\pi/2 < arg\,z < 3\pi/2$$

and consider its integral along the positively oriented contour consisting of the segment $L_1 : z - r,\ \rho \leq r \leq R$, the large semi-circle $C_R : z = R\,exp(i\theta),\ 0\leq\theta\leq\pi$, the segment L_2 where $-L_2$ is parameterized as: $z=-r,\ \rho\leq r\leq R$, and the small semi-circle C_ρ where $-C_\rho$ is parameterized as: $z=\rho\,exp(-i\theta),\ 0\leq\theta\leq\pi$. The function $f(z)$ has only one singularity inside the contour at $z=i$ so that, by the Residue Theorem,

$$\int_\rho^R \frac{(ln\,r)^2}{r^2+1}\,dr + \int_\rho^R \frac{(ln\,r+i\pi)^2}{r^2+1}\,dr = 2i\pi\phi(i) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz, \qquad (4)$$

where

$$\phi(z) = \frac{(log\,z)^2}{z+i},\ \phi(i) = \frac{(i\pi/2)^2}{2i} = i\frac{\pi^2}{8}.$$

Hence,

$$2\int_\rho^R \frac{(ln\,r)^2}{r^2+1}\,dr + i2\pi\int_\rho^R \frac{ln\,r}{r^2+1}\,dr - \pi^2\int_\rho^R \frac{1}{r^2+1}\,dr = -\frac{\pi^3}{4} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Now,

$$|\int_{C_R} f(z)\,dz| \leq \pi\frac{(ln\,R+\pi)^2}{\frac{R^2-1}{R}},\ |\int_{C_\rho} f(z)\,dz| \leq \pi\frac{(ln\,\rho+\pi)^2}{\frac{1-\rho^2}{\rho}}$$

and L'Hospital's rule gives that the limits as $R\to\infty$ and as $\rho\to 0$ are both zero. It now follows from (4) on taking limits that

$$\int_0^\infty \frac{(ln\,r)^2}{r^2+1}\,dr + \int_0^\infty \frac{(ln\,r+i\pi)^2}{r^2+1}\,dr = 2\int_0^\infty \frac{(ln\,r)^2}{r^2+1}\,dr + i2\pi\int_0^\infty \frac{ln\,r}{r^2+1}\,dr - \frac{\pi^3}{2} = -\frac{\pi^3}{4},$$

where we have also used the fact, from an earlier exercise, that

$$\int_0^\infty \frac{1}{r^2+1}\,dr = \frac{\pi}{2}.$$

By equating the real and imaginary parts, we finally obtain

$$\int_0^\infty \frac{(ln\,r)^2}{r^2+1}\,dr = \frac{\pi^3}{8},\ \int_0^\infty \frac{ln\,r}{r^2+1}\,dr = 0.$$

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