Math 206 Complex Calculus– Midterm Exam I
Solutions

Q-1) Find all the cube roots of $-8i$.

Solution:

\[-8i = 8e^{i(-\pi/2+2n\pi)}, \quad n \in \mathbb{Z}\]

\[c_k = 2e^{i(-\pi/6+2k\pi/3)} \quad k = 0, 1, 2.\]

\[c_0 = 2e^{i(-\pi/6)} = \sqrt{3} - i,\]

\[c_1 = 2e^{i(-\pi/6+2\pi/3)} = 2e^{\pi/2} = 2i.\]

\[c_2 = 2e^{i(-\pi/6+4\pi/3)} = 2e^{7\pi/6} = -\sqrt{3} - i.\]

Q-2) Let \(a = (\sqrt{3}i - 1)\frac{e^2}{2}\) and \(b = \left(\frac{5}{4} - i\frac{\pi}{3}\right)\). Calculate the principal value of \(a^b\).

Solution:

\[a = e^2e^{i(2\pi/3)}, \quad a^b = \exp(b \log a) = \exp \left((\frac{5}{4} - i\frac{\pi}{3})(2 + i\frac{2\pi}{3})\right) = \exp \left((\frac{5}{2} + \frac{2\pi^2}{9}) + i\frac{\pi}{6}\right)\]

\[a^b = \left[\exp(\frac{5}{2} + \frac{2\pi^2}{9})\right][\sqrt{3} + i\frac{\pi}{2}].\]

Q-3) Find all values of \(z \in \mathbb{C}\) for which \(\sinh z = i\sqrt{2}\) using

I) \(\sinh z = \sinh x \cos y + i \cosh x \sin y\)

II) \(\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}]\).

Solution:

I) \(\sinh x \cos y = 0\) gives either \(x = 0\) or \(y = (1/2 + n)\pi\) where \(n\) is an integer. Putting \(x = 0\) in the second equation \(\cosh x \sin y = \sqrt{2}\) gives \(\sin y = \sqrt{2}\) which is impossible. So we must have \(y = (1/2 + n)\pi\). Putting this into the second equation now gives \((-1)^n \cosh x = \sqrt{2}\), from where it follows that \(n\) is an even integer since \(\cosh x\) is always positive. Solving \(\sqrt{2} = \cosh x = (e^x + e^{-x})/2\) gives \(x = \pm \ln(1 + \sqrt{2})\). The solution set is then \(z = \pm \ln(1 + \sqrt{2}) + i(1/2 + 2n)\pi\).

II) \(\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}] = \log[i\sqrt{2} \pm i] = \log[(\sqrt{2} \pm 1)e^{i(1/2 + 2n)\pi}]\)

\[= \ln(\sqrt{2} \pm 1) + i(1/2 + 2n)\pi = \pm \ln(\sqrt{2} + 1) + i(1/2 + 2n)\pi.\]
Q-4) Evaluate the integral \[ \int_C \frac{z^2 + z + 1}{z^2(z-1)(z-2)} \, dz \], where \( C \) is the circle with radius 3/2 centered at the origin and oriented positively.

Solution:

Let \( f(z) = \frac{z^2 + z + 1}{z^2(z-1)(z-2)} \) and \( g(z) = \frac{z^2 + z + 1}{z^2} \). Also let \( C_0 \) be a circle of radius 1/3 centered at \( z = 0 \), and \( C_1 \) a circle of radius 1/3 centered at \( z = 1 \). Then

\[
\int_C \frac{z^2 + z + 1}{z^2(z-1)(z-2)} \, dz = \int_{C_0} \frac{f(z)\,dz}{z^2} + \int_{C_1} \frac{g(z)\,dz}{z-1} = 2\pi i \left( f'(0) + g(1) \right) \quad \text{(Cauchy Integral Formula)}
\]

\[
= 2\pi i \left( \frac{5}{4} - 3 \right) = -\frac{7\pi}{2} i.
\]

Q-5) Let \( S \) be the positively oriented square path formed in the complex plane by joining the points 1, \( i \), \(-1\) and \(-i\). Evaluate the integral \[ \frac{1}{\pi i} \int_S \frac{\tan z}{(3z - \pi)^3} \, dz \].

Solution:

The integrand is analytic on and inside the contour of integration and by the Cauchy-Goursat theorem the integral is zero.