Date: 20 March 2004, Saturday
Time: 10:30-12:30

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Math 206 Complex Calculus– Midterm Exam I
Solutions

Q-1) Find all the cube roots of $-8i$.

Solution:

$$\begin{aligned}
-8i &= 8e^{i(-\pi/2+2n\pi)}, \quad n \in \mathbb{Z} \\
c_k &= 2e^{i(-\pi/6+2k\pi/3)} \quad k = 0, 1, 2. \\
c_0 &= 2e^{i(-\pi/6)} = \sqrt{3} - i, \\
c_1 &= 2e^{i(-\pi/6+2\pi/3)} = 2e^{\pi/2} = 2i. \\
c_2 &= 2e^{i(-\pi/6+4\pi/3)} = 2e^{7\pi/6} = -\sqrt{3} - i.
\end{aligned}$$

Q-2) Let $a = \left(\sqrt{3}i - 1\right)\dfrac{e^2}{2}$ and $b = \left(\dfrac{5}{4} - i\dfrac{\pi}{3}\right)$. Calculate the principal value of a^b.

Solution:

$a = e^2 e^{i(2\pi/3)}$, $a^b = \exp(b \log a) = \exp\left((\frac{5}{4} - i\frac{\pi}{3})(2 + i\frac{2\pi}{3})\right) = \exp\left[(\frac{5}{2} + \frac{2\pi^2}{9}) + i\frac{\pi}{6}\right]$
$a^b = [\exp(\frac{5}{2} + \frac{2\pi^2}{9})][\frac{\sqrt{3}}{2} + i\frac{1}{2}]$.

Q-3) Find all values of $z \in \mathbb{C}$ for which $\sinh z = i\sqrt{2}$ using
I) $\sinh z = \sinh x \cos y + i \cosh x \sin y$
II) $\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}]$.

Solution:

I) $\sinh x \cos y = 0$ gives either $x = 0$ or $y = (1/2 + n)\pi$ where n is an integer. Putting $x = 0$ in the second equation $\cosh x \sin y = \sqrt{2}$ gives $\sin y = \sqrt{2}$ which is impossible. So we must have $y = (1/2 + n)\pi$. Putting this into the second equation now gives $(-1)^n \cosh x = \sqrt{2}$, from where it follows that n is an even integer since $\cosh x$ is always positive. Solving $\sqrt{2} = \cosh x = (e^x + e^{-x})/2$ gives $x = \pm\ln(1 + \sqrt{2})$. The solution set is then $z = \pm\ln(1 + \sqrt{2}) + i(1/2 + 2n)\pi$.

II) $\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}] = \log[i\sqrt{2} \pm i] = \log[(\sqrt{2} \pm 1)e^{i(1/2+2n)\pi}]$
$= \ln(\sqrt{2} \pm 1) + i(1/2 + 2n)\pi = \pm\ln(\sqrt{2} + 1) + i(1/2 + 2n)\pi$.

Q-4) Evaluate the integral $\int_C \frac{z^2+z+1}{z^2(z-1)(z-2)}\,dz$, where C is the circle with radius 3/2 centered at the origin and oriented positively.

Solution:

Let $f(z) = \frac{z^2+z+1}{(z-1)(z-2)}$ and $g(z) = \frac{z^2+z+1}{z^2(z-2)}$. Also let C_0 be a circle of radius 1/3 centered at $z = 0$, and C_1 a circle of radius 1/3 centered at $z = 1$. Then

$$\begin{aligned}\int_C \frac{z^2+z+1}{z^2(z-1)(z-2)}\,dz &= \int_{C_0} \frac{f(z)dz}{z^2} + \int_{C_1} \frac{g(z)dz}{z-1} \\ &= 2\pi i\,(f'(0)+g(1)) \quad \text{(Cauchy Integral Formula)} \\ &= 2\pi i\left(\frac{5}{4}-3\right) = -\frac{7}{2}\pi i.\end{aligned}$$

Q-5) Let S be the positively oriented square path formed in the complex plane by joining the points 1, i, -1 and $-i$. Evaluate the integral $\frac{1}{\pi i}\int_S \frac{\tan z}{(3z-\pi)^3}\,dz$.

Solution:

The integrand is analytic on and inside the contour of integration and by the Cauchy-Goursat theorem the integral is zero.
