Date: 1 May 2004, Saturday Instructors: Özgüler & Sertöz
Time: 10:00-12:00

Math 206 Complex Calculus– Midterm Exam II
Solutions

Q-1) Evaluate $\displaystyle\int_0^\infty \frac{x^{1/3}}{(8+x)^3}\,dx$.

Note: If certain limits are used in your solution, show clearly how they are evaluated.

Solution:

Use contour of Figure 70 on page 224 with $0<\rho<8<R$. Let $f(z)=z^{1/3}/(8+z)^3$. Residue of f at $z=-8$ is obtained by evaluating $(1/2)(z^{1/3})''$ at $z=-8$. This gives $Res=-(1/9)(-8)^{-5/3}$. Calculating this using complex log gives Res$=-\dfrac{1}{576}-i\dfrac{\sqrt{3}}{576}$. Along the lower path on the real axis, $z=xe^{2\pi i}$ and since we are travelling backwards the integral gains a multiplicative factor of $-e^{2\pi i/3}$. If I denotes the value of our integral then, after taking limits, we get $(1-e^{2\pi i/3})I=2\pi i$ Res. Solving this we get $I=\dfrac{\pi\sqrt{3}}{432}$.

Q-2) Evaluate $\displaystyle\frac{1}{2\pi i}\int_C \frac{z^7}{1-2z^8}\,dz$, where C is the unit circle traversed counterclockwise.

Solution:

This integral is equal to the sum of the residues of $\frac{z^7}{1-2z^8}$, all of which are inside the unit circle. Its residue at any of its roots is $Res_{z=z_0}\dfrac{z^7}{1-2z^8}=\left.\dfrac{z^7}{(1-2z^8)'}\right|_{z=z_0}=-\dfrac{1}{16}$, and is independent of which root is involved. There are 8 roots, so the sum of the residues is $-\frac{1}{2}$.

We can also use the residue at infinity concept, Theorem 2 on page 185. Then this integral is equal to the residue at $z=0$ of $\dfrac{1}{z^2}\dfrac{(1/z)^7}{1-2(1/z)^8}$. It is easily calculated to be -1/2.

Q-3) Using Laplace transform techniques solve the initial value problem

$$f''(t)-3f'(t)+2f(t)=1, \quad \text{whith} \quad f(0)=0,\ \ f'(0)=1.$$

Solution:

Hitting with Laplace, solving for $F(s)$ and using partial fractions give

$$F(s)=\frac{1}{2s}-\frac{2}{s-1}+\frac{3}{2(s-2)}.$$

Then $f(t)=\dfrac{1}{2}-2\,e^t+\dfrac{3}{2}\,e^{2t}$.

Q-4) Solve the Volterra equation $x(t) = \cos t + \int_0^t \sinh(t-u)\, x(u)\, du$.

Solution:

See page 7 of the notes on Laplace. $X(s) = \dfrac{F(s)}{1-H(s)} = \dfrac{1}{3}\dfrac{s}{s^2-2} + \dfrac{2}{3}\dfrac{s}{s^2+1}$. Then $x(t) = \dfrac{1}{3}\cosh\sqrt{2}\, t + \dfrac{2}{3}\cos t$.

Q-5) Using z-transform techniques solve the recurrence equation $f(n+3) = 2f(n+2) - f(n)$ where $f(0) = 1$, $f(1) = 2$, and $f(2) = 4$.

Solution:

Transforming the given equation with z-transform and solving for $F(z)$ gives $F(z) = \dfrac{z^3}{z^3 - 2z^2 + 1}$. The denominator is easily seen to have $z = 1$ as a root. Using this we get $z^3 - 2z^2 + 1 = (z-1)(z^2 - z - 1)$. Its roots are 1, $\alpha = (1+\sqrt{5})/2$, and $\beta = (1-\sqrt{5})/2$. Then using the residue method to calculate inverse z-transform (see pages 25-26 of the notes) we get $f(n) = \sum Res \dfrac{z^{n+2}}{z^3 - 2z^2 + 1}$. This is easily calculated to be $f(n) = \dfrac{2}{5-\sqrt{5}}\,\alpha^{n+2} + \dfrac{2}{5+\sqrt{5}}\,\beta^{n+2} - 1$, $n = 0, 1, 2, \ldots$.

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