Math 206 Complex Calculus– Midterm Exam II
Solutions

Q-1) Evaluate \( \int_0^\infty \frac{x^{1/3}}{(8 + x)^3} \, dx. \)

Note: If certain limits are used in your solution, show clearly how they are evaluated.

Solution:

Use contour of Figure 70 on page 224 with \( 0 < \rho < 8 < R \). Let \( f(z) = z^{1/3}/(8+z)^3 \). Residue of \( f \) at \( z = -8 \) is obtained by evaluating \((1/2)(z^{1/3})''\) at \( z = -8 \). This gives \( \text{Res} = -(1/9)(-8)^{-5/3} \).

Calculating this using complex log gives \( \text{Res} = -\frac{1}{576} - i\frac{\sqrt{3}}{576} \). Along the lower path on the real axis, \( z = xe^{2\pi i} \) and since we are travelling backwards the integral gains a multiplicative factor of \(-e^{2\pi i/3}\). If \( I \) denotes the value of our integral then, after taking limits, we get

\[
(1 - e^{2\pi i/3})I = 2\pi i \, \text{Res}.
\]

Solving this we get \( I = \frac{\pi \sqrt{3}}{432} \).

Q-2) Evaluate \( \frac{1}{2\pi i} \int_C \frac{z^7}{1-2z^8} \, dz \), where \( C \) is the unit circle traversed counterclockwise.

Solution:

This integral is equal to the sum of the residues of \( \frac{z^7}{1-2z^8} \), all of which are inside the unit circle.

Its residue at any of its roots is \( \text{Res}_{z=z_0} \frac{z^7}{1-2z^8} = \frac{z^7}{(1-2z^8)'} |_{z=z_0} = -\frac{1}{16} \), and is independent of which root is involved. There are 8 roots, so the sum of the residues is \( \frac{1}{8} \).

We can also use the residue at infinity concept, Theorem 2 on page 185. Then this integral is equal to the residue at \( z = 0 \) of \( \frac{1}{z^2} \frac{(1/z)^7}{1-2(1/z)^8} \). It is easily calculated to be \(-1/2\).

Q-3) Using Laplace transform techniques solve the initial value problem

\[
f''(t) - 3f'(t) + 2f(t) = 1, \quad \text{with} \quad f(0) = 0, \quad f'(0) = 1.
\]

Solution:

Hitting with Laplace, solving for \( F(s) \) and using partial fractions give

\[
F(s) = \frac{1}{2s} - \frac{2}{s-1} + \frac{3}{2(s-2)}.
\]

Then \( f(t) = \frac{1}{2} - 2e^t + \frac{3}{2} e^{2t}. \)
Q-4) Solve the Volterra equation \( x(t) = \cos t + \int_0^t \sinh(t-u) \, x(u) \, du \).

Solution:

See page 7 of the notes on Laplace. \( X(s) = \frac{F(s)}{1 - H(s)} = \frac{1}{3} \frac{s}{s^2 - 2} + \frac{2}{3} \frac{s}{s^2 + 1} \). Then \( x(t) = \frac{1}{3} \cosh \sqrt{2} \, t + \frac{2}{3} \cos t \).

Q-5) Using z-transform techniques solve the recurrence equation \( f(n + 3) = 2f(n + 2) - f(n) \)

where \( f(0) = 1 \), \( f(1) = 2 \), and \( f(2) = 4 \).

Solution:

Transforming the given equation with z-transform and solving for \( F(z) \) gives \( F(z) = \frac{z^3}{z^3 - 2z^2 + 1} \).

The denominator is easily seen to have \( z = 1 \) as a root. Using this we get \( z^3 - 2z^2 + 1 = (z-1)(z^2 - z - 1) \). Its roots are \( 1, \alpha = (1 + \sqrt{5})/2, \) and \( \beta = (1 - \sqrt{5})/2. \) Then using the residue method to calculate inverse z-transform (see pages 25-26 of the notes) we get \( f(n) = \sum_{n=0}^{\infty} \text{Res} \frac{z^{n+2}}{z^3 - 2z^2 + 1}. \) This is easily calculated to be \( f(n) = \frac{2}{5 - \sqrt{5}} \alpha^{n+2} + \frac{2}{5 + \sqrt{5}} \beta^{n+2} - 1, \) 

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