Date: 30 April 2005, Saturday NAME:..
Time: 10:00-12:00
Instructors: Özgüler & Sertöz STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet.
No correct answer without a satisfying reasoning is accepted. Show your work in detail.
Write your name on the top of every page.

Q-1) Use Laplace transform techniques to solve the following boundary value problem:

$$f'''(t) - 6f''(t) + 11f'(t) - 6f(t) = -3, \quad f(0) = f'(0) = 0, \quad f''(0) = 1.$$

Solution: The characteristic polynomial is $s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3)$. Moreover the $s-3$ factor cancels while solving for $F(s)$ and we get $F(s) = 1/[s(s-1)(s-2)]$ which is easily dealt with using partial fractions technique to get $f(t) = \frac{1}{2} - e^t + \frac{1}{2}\, e^{2t}$.

Q-2) Use Laplace transform techniques to solve the following system where $x(0) = y(0) = 0$.

$$\begin{aligned} \frac{dx}{dt} - 6x - 3y &= \cosh 6t, \\ 12x + \frac{dy}{dt} + 6y &= e^{6t}. \end{aligned}$$

Solution: Hitting with Laplace and solving for X and Y we easily get $X = \dfrac{s+3}{s^2(s-6)}$ and $Y = \dfrac{s^2 - 12s - 36}{s^2(s-6)(s+6)}$. Using cover up method in the partial fraction process we quickly obtain $X = -\dfrac{1}{4s} - \dfrac{1}{2s^2} + \dfrac{1}{4(s-6)}$ and $Y = \dfrac{1}{3s} + \dfrac{1}{s^2} - \dfrac{1}{6(s-6)} - \dfrac{1}{6(s+6)}$. Each term is now recognizable and Laplace inverse gives the solution as $x = -\dfrac{1}{4} - \dfrac{t}{2} + \dfrac{1}{4}e^{6t}$ and $y = \dfrac{1}{3} + t - \dfrac{1}{3}\cosh 6t$.

Q-3) Use residues method to evaluate $\displaystyle\int_0^\infty \frac{x^{1/3}}{(1+x)^2}\,dx$.

Solution: Use the indented path of figure 70 on page 224 of the sixth edition of Brown & Churchill. The residue of $z^{1/3}/((1+z)^2$ at $z=1$ can be calculated by evaluating $(1/3)z^{-2/3}$ at $z=-1$. It is crucial to write $z^a = \exp(a\log z)$ and $-1=\exp(i\pi)$, where $a=-2/3$. This gives the residue as $-1/6 - i\sqrt{3}/6$. If we denote the above integral by I, then parameterizing $\int \frac{z^{1/3}}{(1+z)^2}\,dz$ on the lower path along the real line and getting the appropriate limits we get $(-1/2 - i\sqrt{3}/2)I$. The integral on the circular paths vanish as the limits are taken and we finally obtain

$$(\frac{1}{2} - i\frac{\sqrt{3}}{3})I = \frac{\sqrt{3}\pi}{3} - i\frac{\pi}{3}.$$

Equating real or imaginary parts of both sides we easily find

$$I = \frac{2\pi\sqrt{3}}{9}.$$

Q-4) Use residues method to evaluate $\displaystyle\int_0^\infty \frac{(\ln x)^2}{(1+x^2)^2}\,dx$.

Hint: $\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = -\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx = \frac{\pi}{4}$.

Solution: Use the path of figure 69 on page 220 of the text book and follow what is done in example 2 on the next page. Use the function $f(z) = (\log z)^2/(1+z^2)^2$. Let $\phi(z) = (\log z)^2/(z+i)^2$. Then the residue of f at $z = i$ is $\phi'(i) = -\dfrac{\pi}{4} + i\dfrac{\pi^2}{16}$.

On L_2, $z = xe^{i\pi}$ and hence $(\log z)^2 = (\ln x + i\pi)^2 = (\ln x)^2 + 2\pi i \ln x - \pi^2$. Thus parameterizing the integral on L_2 we obtain

$$\int_\rho^R \frac{(\ln x)^2}{(1+x^2)^2}\,dx + 2\pi i \int_\rho^R \frac{\ln x}{(1+x^2)^2}\,dx - \pi^2 \int_\rho^R \frac{1}{(1+x^2)^2}\,dx.$$

Add to this the integral on L_1, which is simply $\int_\rho^R \frac{(\ln x)^2}{(1+x^2)^2}\,dx$, take limits observing that on circular paths the contribution is zero, use the hint, to get

$$\int_0^\infty \frac{(\ln x)^2}{(1+x^2)^2}\,dx = \frac{\pi^3}{16}.$$