Math 206 - Homework #1

1. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, prove that
   (a) the equation $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$.
   (b) the equation $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

2. Establish the identity
$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$
   where $z \neq 1$ and then use it to derive Lagrange's trigonometric identity:
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\sin(\theta/2)}, (0 < \theta < 2\pi)$$
   Hint : After proving the first identity, substitute $z = e^{i\theta}$ in it.

3. Use mathematical induction to verify de Moivre's formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
   where n is a positive integer $(n = 1, 2, \ldots)$.

4. Show that if z_0 is any of the n roots of the equation $z^n = 1$, where $z_0 \neq 1$, then
$$1 + z_0 + z_0^2 + z_0^3 + \cdots + z_0^{n-1} = 0$$

5. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.

   Hint : Factor $z^4 + 4 = (z^2 + az + b)(z^2 + cz + d)$ where a, b, c, d are real numbers.