Date: March 11, 2006, Saturday NAME:..
Time: 10:00-12:00

STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam I

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your exam booklet.

No correct answer without a satisfying reasoning is accepted. Show your work in detail.

Write your name on the top of every page.

Q-1) (i) Find all the fourth roots of $-8+i8\sqrt{3}$ and (ii) indicate the principal root.

Solution: (i) The number $z := -8 + i8\sqrt{3}$ has the polar representation $z = 16\, exp[i(\frac{2\pi}{3} + 2n\pi)]$ for integer n. The principal argument of z is $\frac{2\pi}{3}$. Let $c_k,\ k = 0, 1, 2, 3$ be the four fourth roots of z. Then, $c_k = 2\, exp[i(\frac{\pi}{6} + \frac{k\pi}{2})],\ k = 0, 1, 2, 3$. In rectangular representation

$$c_0 = \sqrt{3} + i,\ c_1 = -1 + \sqrt{3},\ c_2 = -\sqrt{3} - i,\ c_3 = 1 - i\sqrt{3}.$$

(ii) The principal root is the one obtained from the principal value of z, i.e., from $z = 16\, exp(i\frac{2\pi}{3})$. Thus, the principal root is $c_0 = \sqrt{3} + i$.

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Q-2) (i) Calculate $(\sqrt{3}+i)^{1+i}$ and (ii) indicate its principal value.

Solution: (i) We have

$$\begin{aligned}
(\sqrt{3}+i)^{1+i} &= exp\{(1+i)[log(\sqrt{3}+i)]\} = exp\{(1+i)[ln\,2 + i(\tfrac{\pi}{6}+2k\pi)]\} \\
&= exp[ln\,2 - \tfrac{\pi}{6} - 2k\pi + i(ln\,2 + \tfrac{\pi}{6} + 2k\pi)] \\
&= exp(ln\,2)\,exp(-\tfrac{\pi}{6} - 2k\pi)\,exp[i(ln\,2 + \tfrac{\pi}{6} + 2k\pi)] \\
&= 2\,exp(-\tfrac{\pi}{6} - 2k\pi)\,exp[i(ln\,2 + \tfrac{\pi}{6})] \\
&= 2\,exp(-\tfrac{\pi}{6} - 2k\pi)\,[cos(ln\,2 + \tfrac{\pi}{6}) + i\,sin(ln\,2 + \tfrac{\pi}{6})],
\end{aligned}$$

for arbitrary integer k.
(ii) The principal value is obtained when $Log(\sqrt{3}+i)$ is used in the first line above. This corresponds to $k=0$ in the last line, i.e., the principal value of $(\sqrt{3}+i)^{1+i}$ is $2\,exp(-\frac{\pi}{6})\,[cos(ln\,2+\frac{\pi}{6}) + i\,sin(ln\,2+\frac{\pi}{6})]$.

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Q-3) Let $u(\lambda; x, y) = (\lambda x + \cos x)\cosh y$, where λ is a real constant. For which value of λ can we find a function $v(x, y)$ such that $f(z) = u(\lambda; x, y) + iv(x, y)$ will be an entire function of $z = x + iy$? For that value of λ find $v(x, y)$.

Solution: First Cauchy-Riemann equation requires that

$$u_x = (\lambda - sin\, x)cosh\, y = v_y,$$

which gives $v = (\lambda - sin\, x)sinh\, y + \phi(x)$ for some differentiable function ϕ of x. By the second Cauchy-Riemann equation

$$v_x = -cos\, x\, sinh\, y + \phi'(x) = -u_y = -(x\lambda + cos\, x)sinh\, y$$

which gives $\phi'(x) = -x\lambda\, sinh\, y$. Since $\phi'(x)$ must be independent of y, this last equality gives $\lambda = 0$. Therefore, $\phi(x) = c$ for some real constant c and

$$v(x, y) = -sin\, x\, sinh\, y + c$$

is a harmonic conjugate of $u(0; x, y)$ for every real c.

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Q-4) Evaluate

$$\int_C \bar{z}^2 \, dz,$$

where C is the parabolic arc $y = x^2, 0 \leq x \leq 1$, directed from the origin to the point $1 + i$.

Solution: We have $f(z) := \bar{z}^2 = (x - iy)^2 = x^2 - y^2 - i2xy$ and on the contour $C : z = z(x) = x + ix^2$, this function becomes $f[z(x)] = x^2 - x^4 - i2x^3$. Also noting that $z'(x) = 1 + i2x$, we can write

$$\begin{aligned} \int_C \bar{z}^2 \, dz &= \int_0^1 (x^2 - x^4 - i2x^3)(1 + i2x) \, dx \\ &= \int_0^1 (x^2 + 3x^4 - i2x^5) \, dx \\ &= (\tfrac{x^3}{3} + 3\tfrac{x^5}{5})|_0^1 - i(\frac{x^6}{3})|_0^1 \\ &= \tfrac{14}{15} - i\tfrac{1}{3}. \end{aligned}$$