Date: April 22, 2006, Saturday NAME:..
Time: 10:00-12:00

STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your exam booklet.

No correct answer without a satisfying reasoning is accepted. Show your work in detail.

Write your name on the top of every page.

Q-1) Find the residues of $f(z) = \left(\sinh \frac{1}{z}\right)^{-1}$ at all of its poles.

Solution: First recall that $\sinh t = 0$ if and only if $t = i\pi n$ for all integers n. Moreover these are simple zeros of $\sinh t$. Therefore $f(z)$ has simple poles at $z_n = 1/(i\pi n)$ for $n = \pm 1, \pm 2, \ldots$. Note that $z = 0$ is not an isolated singularity since $\lim_{n\to\pm\infty} z_n = 0$.

Since $f(z)$ is of the form $\frac{p(z)}{q(z)}$ with $q(z)$ having simple zeroes at each z_n and $p(z_n) \neq 0$, the residue R_n at z_n is equal to $\frac{p(z_n)}{q'(z_n)}$. Here $q'(z) = -\frac{1}{z^2}\cosh(1/z)$. Recalling that $\cosh(i\pi n) = (-1)^n$ we find that $R_n = \frac{(-1)^n}{n^2\pi^2}$.

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Q-2) Calculate the integral $\int_C \frac{dz}{z^2 \sin z}$, where C is the positively oriented circle $|z| = 3\pi/2$.

Solution: Inside this contour we have three singularities at $z = 0, -\pi, \pi$.

Residues at $\pm\pi$ are easy to calculate:

$$\frac{1}{z^2 \sin z} = \frac{1/z^2}{\sin z},$$

so

$$\mathrm{Res}_{z=\pm\pi} = \frac{1/(\pm\pi)^2}{\cos \pm\pi} = -\frac{1}{\pi^2}.$$

Residue at $z = 0$ requires a little more calculation:

$$\begin{aligned}
\frac{1}{z^2 \sin z} &= \frac{1}{z^2 \left(z - z^3/6 + z^5/120 - \cdots\right)} \\
&= \frac{1}{z^3 \left(1 - z^2/6 + z^4/120 - \cdots\right)} \\
&= \frac{1}{z^3 \left(1 - [z^2/6 - z^4/120 + \cdots]\right)} \\
&= \frac{1}{z^3}\left(1 + \left[z^2/6 - z^4/120 + \cdots\right] + \left[z^2/6 - z^4/120 + \cdots\right]^2 + \cdots\right) \\
&= \frac{1}{z^3}\left(1 + z^2/6 + \text{higher degree terms in } z\right) \\
&= \frac{1}{z^3} + \frac{1/6}{z} + \cdots
\end{aligned}$$

So the residue at $z = 0$ is $1/6$.

Putting these together we find

$$\int_{|z|=3\pi/2} \frac{dz}{z^2 \sin z} = 2\pi i \left(-\frac{2}{\pi^2} + \frac{1}{6}\right).$$

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Q-3) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx.$

Solution: Let $f(z) = \dfrac{z^2}{z^4+z^2+1}$.

The roots of the denominator that lie in the upper half plane are $z_1 = e^{i\pi/3}$ and $z_2 = e^{i2\pi/3}$.

$Res_{z=z_1} f(z) = \dfrac{3-i\sqrt{3}}{12}$ and $Res_{z=z_2} f(z) = -\dfrac{3+i\sqrt{3}}{12}$.

Hence the sum of the residues is $-\dfrac{i}{2\sqrt{3}}$.

Consider the path $\gamma_R = [-R, R] + C_R$ where $R > 1$ and C_R is $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi$.

It can be shown that $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+z^2+1}\,dz = 0.$

From $\displaystyle\int_{\gamma_R} f(z)\,dz = 2\pi i\left(-\frac{i}{2\sqrt{3}}\right)$, it follows that

$$\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx = \frac{\pi}{2\sqrt{3}}.$$

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Q-4) Evaluate the integral $\displaystyle\int_0^\infty \frac{\sin(2x)}{x(x^2+5)}\,dx.$

Solution: Consider the function $f(z) = \dfrac{e^{i2z}}{z(z^2+5)} = \dfrac{e^{i2z}}{z^3+5z}.$

Integrate this function over the path $\gamma_R = [-R,-\rho] + C_\rho + [\rho, R] + C_R$ where $0 < \rho < \sqrt{5} < R$, C_R is $z = Re^{i\theta}$ with $0 \le \theta \le \pi$ and C_ρ is $z = \rho e^{i(\pi-\theta)}$ with $0 \le \theta \le \pi$.

There is a simple pole of $f(z)$ inside the contour at $z_0 = i\sqrt{5}$, and the residue there is $\dfrac{e^{i2z_0}}{3z_0^2+5} = -\dfrac{e^{-2\sqrt{5}}}{10}.$

There is a simple pole at $z = 0$ and the residue there is $B_0 = \left.\dfrac{e^{i2z}}{3z^2+5}\right|_{z=0} = \dfrac{1}{5}.$

We know that $\displaystyle\int_{C_\rho} f(z)\,dz = -\pi i\, B_0 = -\frac{\pi i}{5}.$

We also have $\displaystyle\int_{-\rho}^{-R} f(z)\,dz + \int_\rho^R f(z)\,dz = 2i\int_\rho^r \frac{\sin(2x)}{x(x^2+5)}\,dx.$

The residue theorem now gives

$$\int_{\gamma_R} f(z)\;dz = \int_{C_R} f(z)\;dz + \int_{C_\rho} f(z)\;dz + 2i\int_\rho^R \frac{x\sin(2x)}{x^2+5}\;dx = 2\pi i\left(-\frac{e^{-2\sqrt{5}}}{10}\right).$$

It can be shown by using Jordan's lemma that the integral over C_R vanishes as R tends to infinity.

This finally gives, after taking limits as $\rho \to 0$ and $R \to \infty$,

$$\int_0^\infty \frac{\sin(2x)}{x(x^2+5)}\;dx = \frac{\pi}{10}(1 - e^{-2\sqrt{5}}) \approx 0.3105706584.$$