Date: May 18, 2007, Friday NAME:..

Time: 9:00-11:00

Özgüler & Sertöz STUDENT NO:..

Math 206 Complex Calculus – Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) Evaluate the integral $\displaystyle\int_R \frac{\cot z}{z^4}\, dz$, where R is the positively oriented boundary of the rectangle whose corners are at the points $2+4i$, $-2+4i$, $-2-4i$ and $2-4i$.

Solution: There is only one pole at $z=0$ in this region. The value of the integral is then equal to $2\pi i$ times the residue of $\dfrac{\cot z}{z^4}$ at $z=0$. We first find this residue:

$$\begin{aligned}
\frac{\cot z}{z^4} &= \frac{\cos z}{(\sin z)(z^4)} \\
&= \frac{\cos z}{(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots)(z^4)} \\
&= \frac{\cos z}{(1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots)(z^5)} \\
&= \frac{(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots)(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots)}{z^5} \\
&= \left(\cdots - \frac{z^4}{45} + \cdots\right)\frac{1}{z^5}
\end{aligned}$$

from where we see that the residue is $-\dfrac{1}{45}$.

Hence the value of the integral is $-\dfrac{2\pi i}{45}$.

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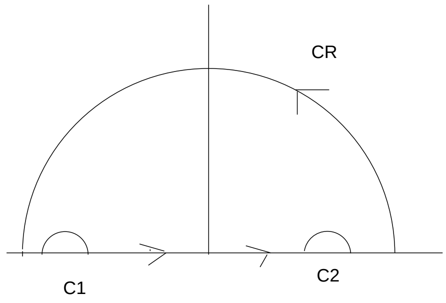
Q-2) Evaluate the integral $P.V. \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2-1)}$.

Observe that there are singularities on the real axis. Avoid these singularities by some semicircles.

Solution: Use the path



together with the function $f(z) = \frac{1}{(z^2+1)(z^2-1)}$ which has simple poles.

In the above figure;

C_R is $Re^{i\theta}$, $0 \le \theta \le \pi$, with $R > 2$.

$-C_1$ is $-1 + \rho e^{i\theta}$, $0 \le \theta \le \pi$, with $0 < \rho < 1$.

$-C_2$ is $1 + \rho e^{i\theta}$, $0 \le \theta \le \pi$, with $0 < \rho < 1$.

The integral of f over the above path is $C = (2\pi i)\text{Res}_{z=i} f(z) = (2\pi i)\frac{i}{4} = -\frac{\pi}{2}$.

The integral on C_R vanishes as R goes to infinity.

The integral on C_1 is $A = (-\pi i)\text{Res}_{z=-1} f(z) = (-\pi i)\frac{-1}{4} = \frac{\pi i}{4}$ as ρ goes to zero.

The integral on C_2 is $B = (-\pi i)\text{Res}_{z=1} f(z) = (-\pi i)\frac{1}{4} = -\frac{\pi i}{4}$ as ρ goes to zero.

Finally taking limits as R goes to infinity and ρ goes to zero, we get

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2-1)} + A + B = C$$

which gives

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2-1)} = -\frac{\pi}{2}.$$

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Q-3) Using the $\mathcal{Z}$-transform method, solve the initial value difference problem

$$f(n+2) - 4f(n+1) + 5f(n) = 0, \quad f(0) = 1, \quad f(1) = 5.$$

Solution: Applying $\mathcal{Z}$-transform to both sides of the equation we obtain

$$F(z) = \frac{z^2 + z}{z^2 - 4z + 5}.$$

To find the inverse we can use both the partial fractions method and the residue method. Here they are:

1) Partial Fractions Method: Let θ be the acute angle with $\tan\theta = \frac{1}{2}$. Then $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$. Let $c = \sqrt{5}$:

$$\begin{aligned}
\frac{z^2+z}{z^2-4z+5} &= \frac{\left(\frac{z}{c}\right)^2 + \frac{1}{c}\left(\frac{z}{c}\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{2}{c}\right)\left(\frac{z}{c}\right) + 1} \\
&= \frac{\left(\frac{z}{c}\right)^2 - \frac{2}{c}\left(\frac{z}{c}\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{2}{c}\right)\left(\frac{z}{c}\right) + 1} + 3\,\frac{\frac{1}{c}\left(\frac{z}{c}\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{2}{c}\right)\left(\frac{z}{c}\right) + 1} \\
&= \frac{\left(\frac{z}{c}\right)^2 - (\cos\theta)\left(\frac{z}{c}\right)}{\left(\frac{z}{c}\right)^2 - 2(\cos\theta)\left(\frac{z}{c}\right) + 1} + 3\,\frac{(\sin\theta)\left(\frac{z}{c}\right)}{\left(\frac{z}{c}\right)^2 - 2(\cos\theta)\left(\frac{z}{c}\right) + 1} \\
&= \mathcal{Z}(c^n \cos n\theta) + 3\,\mathcal{Z}(c^n \sin n\theta).
\end{aligned}$$

2) Residues method: $f(n)$ is the sum of the residues of the fraction $\frac{z^{n-1}(z^2+z)}{z^2-4z+5} = \frac{z^n(z+1)}{(z-\alpha)(z-\bar{\alpha})}$, where $\alpha = 2 + i$.

The sum of the residues is $\frac{\alpha^n(3+i)}{2i} - \frac{\bar{\alpha}^n(3-i)}{2i} = \mathrm{Im}\ \alpha^{n+1} + \mathrm{Im}\ \alpha^n = \mathrm{Re}\ \alpha^n + 3\,\mathrm{Im}\ \alpha^n$. Since $\alpha = \sqrt{5}\exp(i\theta)$, where θ is the acute angle whose tangent is $1/2$, we get the answer as

$$f(n) = 5^{n/2}\left(\cos n\theta + 3\sin n\theta\right).$$

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Q-4) Find a bounded harmonic function $H(x, y)$ defined in the upper half plane such that

$$H(x,0) = \begin{cases} a & \text{when } |x| < 1 \\ b & \text{when } |x| > 1 \end{cases}$$

where $0 < a < b$ are fixed real numbers.

Solution: This is a variation of the problem treated in section 101 of the textbook, seventh edition page 363-364.

Consider the mapping $w = \log \dfrac{z-1}{z+1} = \ln \left| \dfrac{z-1}{z+1} \right| + i \arg \left(\dfrac{z-1}{z+1} \right) = u + iv.$

This maps the upper half plane onto the horizontal strip $0 \leq v \leq \pi$. The boundary conditions on the z-plane translate here as b when $v = 0$, and a when $v = \pi$. The function

$$\frac{(a-b)}{\pi} v + b$$

is harmonic in this strip and satisfies the required boundary conditions. Hence the solution is

$$H(x,y) = \frac{(a-b)}{\pi} \arg \left(\frac{z-1}{z+1} \right) + b = \frac{(a-b)}{\pi} \arctan \left(\frac{2y}{x^2 + y^2 - 1} \right) + b,$$

where $0 \leq \arctan t \leq \pi$ is used.