Date: April 21, 2007, Saturday NAME:..
Time: 14:00-16:00
Özgüler & Sertöz STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) Evaluate the integral $\displaystyle\int_0^\infty \frac{x\ln x}{x^3+1}\,dx$.

Hint: You may use the fact that $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx = \frac{2\pi\sqrt{3}}{9}$.

Answer: Let $\alpha = e^{i\pi/3}$, choose constants $0 < \rho < 1 < R$, and consider the path $\gamma = L_1 + C_R + L_2 + C_\rho$ in $\mathbb{C}$ where;
$z \in L_1$ means $z = x$ for $\rho \le x \le R$,
$z \in C_R$ means $z = Re^{i\theta}$ for $0 \le \theta \le 2\pi/3$,
$z \in -L_2$ means $z = \alpha^2 x$ for $\rho \le x \le R$, and
$z \in -C_\rho$ means $z = \rho e^{i\theta}$ for $0 \le \theta \le 2\pi/3$.

Let $f(z) = \dfrac{z\log z}{z^3+1}$ where we use the branch $-\pi/2 \le \theta < 3\pi/2$ for the log function so that it agrees with the real ln function of the original integral. The function $f(z)$ has a simple pole at $z=\alpha$ inside the contour γ.

We easily calculate

$$2\pi i\; Res_{z=\alpha} f(z) = (2\pi i)\frac{\alpha\log\alpha}{3\alpha^2} = (2\pi i)\left(\frac{\pi\sqrt{3}}{18} + i\frac{\pi}{18}\right) = -\frac{\pi^2}{9} + i\frac{\pi^2\sqrt{3}}{9}.$$

By the usual analysis we show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0, \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

By using the above parametrization we find

$$\int_{L_1} f(z)\,dz = \int_\rho^R \frac{x\ln x}{x^3+1}\,dx$$

and

$$\int_{L_2} f(z)\,dz = -(\alpha^4 \log \alpha^2) \int_\rho^R \frac{x}{x^3+1}\,dx - \alpha^4 \int_\rho^R \frac{x \ln x}{x^3+1}\,dx.$$

We finally have

$$\begin{aligned}
\int_\gamma f(z)\,dz &= 2\pi i \; Res_{z=\alpha} f(z) \\
\int_{L_1} f + \int_{C_R} f + \int_{L_2} f + \int_{C_\rho} f &= -\frac{\pi^2}{9} + i\frac{\pi^2\sqrt{3}}{9}.
\end{aligned}$$

Taking limits as $\rho \to 0$ and $R \to \infty$, and using the hint we get

$$\begin{aligned}
(1-\alpha^4) \int_0^\infty \frac{x \ln x}{x^3+1}\,dx - (\alpha^4 \log \alpha^2) \int_0^\infty \frac{x}{x^3+1}\,dx &= -\frac{\pi^2}{9} + i\frac{\pi^2\sqrt{3}}{9} \\
\left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{x \ln x}{x^3+1}\,dx - \frac{2\pi^2}{9} + i\frac{2\pi^2\sqrt{3}}{27} &= -\frac{\pi^2}{9} + i\frac{\pi^2\sqrt{3}}{9}.
\end{aligned}$$

From this, equating the real or imaginary part of the right hand side with that of the left hand side we find

$$\int_0^\infty \frac{x \ln x}{x^3+1}\,dx = \frac{2\pi^2}{27}.$$

NAME: STUDENT NO:

Q-2) Evaluate the integral $\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx$.

Answer: This solution closely follows the calculation of $\int_0^\infty \frac{\sin x}{x}\,dx$ on page 269-270 of the textbook, seventh edition. Here we mark only the deviations from the book.

Our function is $\frac{e^{iz}}{z(z^2+1)}$.
It has a simple pole at $z = i$ inside the given path with residue $-\frac{1}{2e}$.

Hence the right hand side of the first equality in the solution is now $2\pi i(-\frac{1}{2e}) = -\frac{\pi}{e}i$.

The rest of the calculations are similar. Notice that the integral on C_ρ also converges to $-\pi i$.

Putting these together we get

$$2i\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx - \pi i = -\frac{\pi}{e}i$$

which gives us

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi}{2}(1-\frac{1}{e}) = 0.9929326520...$$

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Q-3) Using the Laplace transforms method solve the initial value problem

$$x'' + 2x' + x(t) = H(t) - H(t-1),$$
$$x(0) = 1, x'(0) = -1,$$

where $H(t)$ is the unit step function.

Answer:

$$X(s) = \frac{s^2+s+1}{s(s+1)^2} - \frac{e^{-s}}{s(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)^2} - e^{-s}[\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}]$$

so that $x(t) = 1 - te^{-t} - [1 - e^{-(t-1)} - (t-1)e^{-(t-1)}]H(t-1), t \geq 0.$

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Q-4) Using the Laplace transforms method determine the solution to the system of equations

$$x'' - y(t) = 0,$$
$$y'' + 8\,y(t) + 16\,x(t) = 0,$$
$$x(0) = 0, x'(0) = 0, y(0) = 0, y'(0) = -1.$$

Answer:

$$X(s) = \frac{1}{(s^2 + 2^2)^2}$$

so that using either the method of residues or recognizing that

$$\frac{8}{(s^2+4)^2} = \frac{d}{ds}\frac{s}{s^2+4} + \frac{1}{s^2+4}$$

it follows that $x(t) = -\frac{1}{16}\sin(2t) + \frac{t}{8}\cos(2t)$.

Since $y(t) = x''(t)$ we easily find $y(t) = -\frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)$.