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Time: 17:30-19:30
Altıntaş & Sertöz

Math 206 Complex Calculus – Midterm Exam I – Solutions

Q-1) Find all the roots of $(\frac{z}{z+1})^4+8+8\sqrt{3}\, i=0$. Write the resulting numbers in rectangular form, i.e. of the form $a+ib$ with a and b being real numbers.

Answer: Call $\alpha=\frac{z}{z+1}$. Then $\alpha^4=-8-8\sqrt{3}i$.

$$\alpha^4 = 2^4\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\, i\right) = 2^4\ e^{(4\pi/3+2n\pi)i}.$$

Then every 4-th root of α is of the form $2e^{(\pi/3+n\pi/2)i}$ for $n=0,1,2,3$.

This gives the numbers $\pm(1+i\sqrt{3})$ and $\pm(-\sqrt{3}+i)$.

From the above equation $z=\dfrac{\alpha}{1-\alpha}=\dfrac{\alpha-|\alpha|^2}{1+|\alpha|^2-2\operatorname{Re}\alpha}$; therefore corresponding z values are:

$$z_0=-1+\frac{i}{\sqrt{3}};$$

$$z_1=-\frac{5}{7}-i\frac{\sqrt{3}}{7};$$

$$z_2=-\frac{4+\sqrt{3}-i}{5+2\sqrt{3}};$$

$$z_3=-\frac{-4+\sqrt{3}-i}{-5+2\sqrt{3}};$$

Q-2) Find a complex differentiable function $f(z)=u+iv$, whose imaginary part is $v(x,y)=x^3-3xy^2-2y$ and satisfies $f(0)=1$. Write $f(z)$ as a function of z and not as a function of x and y.

Answer: We impose the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$. Solving for u we find $u=-3x^2y+y^3-2x+C$. Imposing the condition $f(0)=1$ gives $f(x,y)=(-3x^2y+y^3-2x+1)+i(x^3-3xy^2-2y)$. From the uniqueness of analytic functions, this is the only function generalizing $f(x,0)=-2x+1+ix^3$. But clearly $-2z+1+iz^3$ is such a function. So $f(z)=-2z+1+iz^3$.

Q-3) In the z plane two sets are given by

$$A : Im\ z \geq 0$$

$$B : |z+3| > 2|z-3|$$

a) Sketch $A \cap B$ (intersection of A and B).
b) State whether $A \cap B$ is open, closed or neither.

Answer: The points in the set B satisfy $(x+3)^2 + y^2 > 4[(x-3)^2 + y^2]$. Simple algebra yields $x^2 + y^2 - 10x + 9 < 0$. Recombining the terms, we get $(x-5)^2 + y^2 < 16$. This yields inside an open disk around $x = 5$ with radius 4. The set A is a closed set containing the upper half space of the complex plane including the real axis. The intersection is the set containing the upper half of the open disk together with its diameter. The diameter points are part of the boundary points of $A \cap B$ and are included in $A \cap B$. So $A \cap B$ cannot be open. But on the other hand, the boundary points of the disk, i.e. the points lying on the circle $(x-5)^2 + y^2 = 16$ are not included in $A \cap B$, so $A \cap B$ cannot be closed. Hence $A \cap B$ is neither open nor closed.

Q-4) Calculate the principal value of A^B, where $A = 1 - i$ and $B = 2e^{i\pi/6}$. Give your answer in rectangular form.

Answer: $A = 1 - i = \sqrt{2}e^{(-i\pi/4)}$, $\log A = \ln\sqrt{2} - i\pi/4$, and $B = 2e^{i\pi/6} = \sqrt{3} + i$.

Then $A^B = e^{(B \log A)} = e^{[\sqrt{3}+i][\ln\sqrt{2} - i\pi/4]}$, and this gives in rectangular form

$$e^{\sqrt{3}\ln\sqrt{2}+\pi/4}\cos\left(\ln\sqrt{2} - \sqrt{3}\pi/4\right) + ie^{\sqrt{3}\ln\sqrt{2}+\pi/4}\sin\left(\ln\sqrt{2} - \sqrt{3}\pi/4\right).$$

Comments to `sertoz@bilkent.edu.tr` or to `altintas@ee.bilkent.edu.tr`
