Lecture Notes on Laplace and z-transforms

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1 Introduction

These notes are intended to guide the student through problem solving using Laplace and z-transform techniques and is intended to be part of MATH 206 course. These notes are freely composed from the sources given in the bibliography and are being constantly improved. Check the date above to see if this is a new version.

You are welcome to contact me through e-mail if you have any comments on these notes such as praise, criticism or suggestions for further improvements.

2 Laplace Transformation

The main application of Laplace transformation for us will be solving some differential equations. A differential equation will be transformed by Laplace transformation into an algebraic equation which will be solvable, and that solution will be transformed back to give the actual solution of the DE we started with.

We define the **Laplace Transform** of a function $f : [0, \infty) \to \mathbb{C}$ as

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \quad \text{for } s \in \mathbb{C}$$

We sometimes use $F(s)$ to denote $\mathcal{L}(f(t))$ if there is no confusion. But beware of conflicting notation in the literature.

Euler[1] was the first one to use this transformation to solve certain differential equations in 1737. Later Laplace[2] independently used it in his book *Théorie Analytique de Probabilités* in 1812, [6, p285].

2.1 Existence of Laplace Transformation

It is clear that $\mathcal{L}(f)$ does not exist for every function f. For example it can be easily verified that $\mathcal{L}(e^{t^2})$ does not exist, i.e. the associated integral clearly diverges. However $\mathcal{L}$ exists for a large class of functions. For example consider the following class of functions:

A function $f : [0, \infty) \to \mathbb{C}$ is said to be of **exponential order** a if there are positive real constants M, T and a such that $|f(t)| \le Me^{at}$ for all $t \ge T$.

$\mathcal{L}(f)$ exists if f is integrable on $[0, b]$ for every $b > 0$ and f is of exponential order a for some $a > 0$. In this case $F(s)$ is defined if and only if $\operatorname{Re} s > a$. Moreover observe from the definition that $\lim\limits_{\operatorname{Re} s \to \infty} F(s) = 0$.

A word of relief: We will basically be using Laplace transform techniques to

[1]Leonhard Euler 1707-1783.
[2]Pierre-Simon Laplace 1749-1827.

solve differential equations. Most differential equations with initial values will have a unique solution, see for example [7, p498-Thm 10.6 and p501-Thm 10.8]. We therefore formally apply Laplace transform techniques, without checking for validity, and if in the end the function we find solves the differential equation then it is *the* solution. For this reasons most tables of Laplace transforms do not give the range of validity and are therefore wrong per se but perfectly acceptable given the overall purpose.

2.2 Elementary Properties of Laplace Transformation

Before we start calculating the Laplace transformation of any function we can derive some results which reflect our expectations from $\mathfrak{L}(f)$ using only the elementary properties of integrals.

Suppose $\alpha, \beta \in \mathbb{C}$ and f, g functions for which Laplace transformation exists. Then:

- $\mathfrak{L}(\alpha f(t) + \beta g(t)) = \alpha F(s) + \beta G(s)$. (Linearity)
- $\mathfrak{L}(e^{\alpha t} f(t)) = F(s - \alpha)$. (Shift property)
- Suppose f and all its derivatives up to and including order n are continuous on $[0, \infty)$ with f and each derivative having Laplace transformation. Then

  $$\mathfrak{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - \cdots - s^{n-k-1} f^{(k)}(0) - \cdots - f^{(n-1)}(0).$$

  In particular

  $$\begin{aligned} \mathfrak{L}(f'(t)) &= sF(s) - f(0), \\ \mathfrak{L}(f''(t)) &= s^2 F(s) - sf(0) - f'(0). \end{aligned}$$

- If f is continuous on $(0, \infty)$, then $\mathfrak{L}\left(\int_0^t f(u)du\right) = F(s)/s$.
- If f is continuous on $(0, \infty)$, then $\mathfrak{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$.

2.3 Transforms of some elementary functions

Before we apply Laplace transformation techniques to differential equations we need to actually see the transformation of some functions. We generally need some tables listing the Laplace transforms of some elementary functions. Then using the properties listed in the previous section we can find the Laplace transformation of most functions.

We begin by a table where each entry can be found by direct integration, using the definition of the Laplace transformation.

In the following list, α and β are complex constants and n is a nonnegative integer.

- $\mathcal{L}(\alpha) = \dfrac{\alpha}{s}$.
- $\mathcal{L}(t) = \dfrac{1}{s^2}$.
  In general $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$.
- $\mathcal{L}(e^{\alpha t}) = \dfrac{1}{s-\alpha}$, where $\operatorname{Re} s > \operatorname{Re} \alpha$.
- $\mathcal{L}(\sin \alpha t) = \dfrac{\alpha}{s^2+\alpha^2}$, where $\operatorname{Re} s > -\operatorname{Im} \alpha$.
- $\mathcal{L}(\cos \alpha t) = \dfrac{s}{s^2+\alpha^2}$, where $\operatorname{Re} s > -\operatorname{Im} \alpha$.
- $\mathcal{L}(\sinh \alpha t) = \dfrac{\alpha}{s^2-\alpha^2}$.
- $\mathcal{L}(\cosh \alpha t) = \dfrac{s}{s^2-\alpha^2}$.

The next three formulas follow from the general property $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$.

- $\mathcal{L}(t \sin \alpha t) = \dfrac{2\alpha s}{(s^2+\alpha^2)^2}$.

- $\mathcal{L}(t\cos\alpha t) = \dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$.
- $\mathcal{L}(te^{-\alpha t}) = \dfrac{1}{(s+\alpha)^2}$, where $\alpha > 0$.

  In general $\mathcal{L}(t^n e^{-\alpha t}) = \dfrac{n!}{(s+\alpha)^{n+1}}$, where $\alpha > 0$.

The next formulas follow from the shift property $\mathcal{L}(e^{\alpha t}f(t)) = F(s-\alpha)$.

- $\mathcal{L}(e^{-\alpha t}\sin\beta t) = \dfrac{\beta}{(s+\alpha)^2+\beta^2}$, where $\alpha > 0$.
- $\mathcal{L}(e^{-\alpha t}\sinh\beta t) = \dfrac{\beta}{(s+\alpha)^2-\beta^2}$, where $\alpha > 0$.
- $\mathcal{L}(e^{-\alpha t}\cos\beta t) = \dfrac{s+\alpha}{(s+\alpha)^2+\beta^2}$, where $\alpha > 0$.
- $\mathcal{L}(e^{-\alpha t}\cosh\beta t) = \dfrac{s+\alpha}{(s+\alpha)^2-\beta^2}$, where $\alpha > 0$.

2.4 Inverse Laplace Transformation

If $\mathcal{L}(f(t)) = F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F(s)) = f(t)$.

If we assume that the functions whose Laplace transforms exist are going to be taken as continuous then no two different functions can have the same Laplace transform. Functions that differ only at isolated points can have the same Laplace transform. Such uniqueness theorems allow us to find inverse Laplace transform by looking at Laplace transform tables.

Example:-2.1 Find the function $f(t)$ for which $\mathcal{L}(f(t)) = \dfrac{2s+3}{s^2+4s+13}$.
Solution: By completing the denominator to a square and playing with the numerator we write $\mathcal{L}(f(t))$ as

$$\frac{2s+3}{s^2+4s+13} = \frac{2(s+2)}{(s+2)^2+9} - \frac{1}{(s+2)^2+9}.$$

Here we try to recognize each part on the right as Laplace transform of some function, using a table of Laplace transforms. For example we note that $\mathcal{L}(e^{-2t}\cos(3t)) = \frac{s+2}{(s+2)^2+9}$ and $\mathcal{L}(e^{-2t}\sin(3t)) = \frac{3}{(s+2)^2+9}$. Using this information together with the fact that Laplace transform is a linear operator we find that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\} \\
&= 2e^{-2t}\cos(3t) - \frac{1}{3}e^{-2t}\sin(3t) \\
&= f(t).
\end{aligned}$$

Note: Inverse Laplace of a function can also be found using integrals and residues. This is given in your textbook [3, sections 66-67].

2.5 Convolution

When $f(t)$ and $g(t)$ are defined for $t > 0$, and are piecewise continuous, then their convolution, denoted by $f * g$, is defined as

$$(f * g)(t) = \int_0^t f(t-u)g(u)du, \quad \text{for } 0 \leq t < \infty.$$

Convolution has some immediate properties following from the above definition:
1. $f * g = g * f$.
2. $f * (cg) = (cf) * g = c(f * g)$, where c is a constant.
3. $f * (g + h) = f * g + f * h$.
4. $f * (g * h) = (f * g) * h$.

In particular the following property is useful:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \text{ where } \mathcal{L}(f) = F \text{ and } \mathcal{L}(g) = G.$$

In other words;

$$\mathcal{L}((f * g)(t)) = F(s)G(s).$$

Example:-2.2 An equation of the form

$$x(t) = f(t) + \int_0^t h(t-u)x(u)du$$

where f and h are known functions and x is the unknown function is called Volterra[3] integral equation. Note that the given integral is a convolution integral. Letting capital letters denote the Laplace transform of the corresponding function we apply Laplace operator to each side of the Volterra equation to obtain

$$X(s) = F(s) + H(s)X(s).$$

Solving for $X(s)$ we get

$$X(s) = \frac{F(s)}{1 - H(s)},$$

which can theoretically be inverted by Laplace transformation to give the required $x(t)$.

Example:-2.3 Solve the Volterra equation

$$x(t) = e^{-t} - 4\int_0^t \cos 2(t-u)x(u)du.$$

Solution: Applying Laplace operator to each side we get

$$X(s) = \frac{1}{s+1} - 4X(s)\frac{s}{s^2+4}.$$

Solving for $X(s)$ we get

$$\begin{aligned} X(s) &= \frac{s^2+4}{(s+1)(s+2)^2} \\ &= \frac{5}{s+1} - \frac{4}{s+2} + \frac{8}{(s+2)^2}. \end{aligned}$$

Applying Laplace inverse transformation to both sides of this equation we finally get

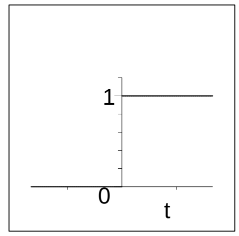
$$x(s) = 5e^{-t} - 4e^{-2t} - 8te^{-2t}.$$

[3]Vito Volterra 1860-1940.

2.6 Heaviside unit function

The Heaviside unit function is denoted and defined as

$$H(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t \geq 0. \end{cases}$$



By directly integrating the Heaviside[4] function we find that

$$\mathcal{L}(H(t-a)) = \frac{e^{-sa}}{s} \quad \text{for } a > 0.$$

In particular

$$\mathcal{L}(H(t)) = \frac{1}{s}.$$

Compare this to the case where we apply Laplace operator to $f(t) = 1$ for $t > 0$.

Again by direct integration we find the important shift property

$$\mathcal{L}(H(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t)), \quad \text{for } a > 0.$$

Example:-2.4

$$\mathcal{L}(H(t-2)\cos(t-2)) = \frac{se^{-2s}}{s^2+1}.$$

Example:-2.5

$$\mathcal{L}(H(t-2)\sin(t-2)) = \frac{e^{-2s}}{s^2+1}.$$

Example:-2.6

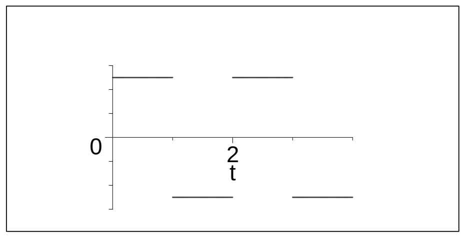
$$\begin{aligned} \mathcal{L}(H(t-2)\cos(t)) &= \mathcal{L}(H(t-2)\cos((t-2)+2)) \\ &= \mathcal{L}(H(t-2)(\cos(t-2)\cos(2) - \sin(t-2)\sin(2))) \\ &= \cos(2)\mathcal{L}(H(t-2)\cos(t-2)) - \sin(2)\mathcal{L}(H(t-2)\sin(t-2)) \\ &= \left(\frac{s\cos(2)}{(s^2+1)} - \frac{\sin(2)}{(s^2+1)}\right)e^{-2s}. \end{aligned}$$

[4] Oliver Heaviside 1850-1925.

2.7 The square wave function

The square wave function, for some $a > 0$, is defined as

$$f(t) = \begin{cases} 1, & \text{for } 2na \le t < (2n+1)a,\ n \in \mathbb{N}; \\ -1, & \text{for } (2n+1)a \le t < (2n+2)a,\ n \in \mathbb{N}. \end{cases}$$



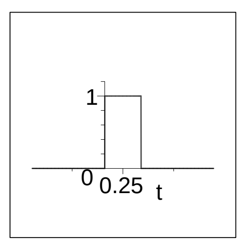
It follows that $f(t) = H(t) + 2\sum_{n=1}^{\infty}(-1)^n H(t - na)$ and consequently

$$\begin{aligned} \mathcal{L}(f(t)) &= \frac{1}{s}(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-nas}) \\ &= \frac{1}{s}\left(1 - \frac{2e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) \\ &= \frac{1}{s}\tanh\left(\frac{as}{2}\right). \end{aligned}$$

2.8 Impulse function

Define a function $f_k(t)$ for some positive number k as follows:

$$f_k(t) = \begin{cases} \frac{1}{k}, & \text{for } 0 \le t \le k; \\ 0, & \text{for } t > k. \end{cases}$$



Note that the area under the graph of f_k is 1. If we take the limit of f_k as k goes to zero, we end up with a function which is zero when $t \neq 0$ and has infinite height at 0, but still with total area 1 under the graph, since it is the limiting position of graphs with area 1. We denote this new function by $\delta(t)$

and call it the impulse function or Dirac[5] delta function.

To find the Laplace transform of the impulse function we start with the Laplace of f_k:

$$\begin{aligned}
\mathcal{L}(f_k(t)) &= \int_0^\infty f_k(t)e^{-st}dt \\
&= \int_0^k \frac{1}{k}e^{-st}dt = \left[-\frac{e^{-st}}{sk}\right]_0^k \\
&= -\frac{1}{sk}(e^{-sk}-1) = 1 - \frac{sk}{2!} + \frac{(sk)^2}{3!} + \cdots .
\end{aligned}$$

Taking the limit as $k \to 0$ we find

$$\mathcal{L}(\delta(t)) = 1.$$

An impulse of size a is represented by $a\delta(t)$ and an impulse which is delayed by time T is denoted by $\delta(t-T)$. Recalling the shift property, i.e. $\mathcal{L}(H(t-T)f(t-T)) = e^{-sT}\mathcal{L}(f(t))$, we can immediately write the Laplace of a delayed impulse function of a certain size:

$$\mathcal{L}(a\delta(t-T)) = a\mathcal{L}(H(t-T)\delta(t-T)) = ae^{-sT}.$$

Example:-2.7 Solve the initial value problem

$$x''(t) + 3x'(t) + 2x(t) = 5\delta(t-2)$$

where $x(0) = 4$ and $x'(0) = 0$.

Solution: Transforming by Laplace we get
$(s^2+3s+2)X(s) - 4s - 12 = 5e^{-2s}$. Solving for $X(s)$ we find

$$X(s) = \frac{5e^{-2s}+4s+12}{(s+1)(s+2)}.$$

Here we observe that

$$\begin{aligned}
\frac{1}{(s+1)(s+2)} &= \frac{1}{s+1} - \frac{1}{s+2}, \\
\frac{s}{(s+1)(s+2)} &= \frac{-1}{s+1} + \frac{2}{s+2}
\end{aligned}$$

[5] Paul Adrien Maurice Dirac 1902-1984.

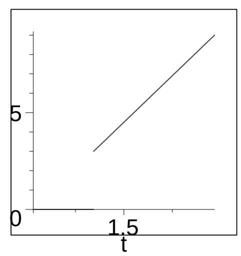
and recall the formulas

$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}, \quad \text{and}$$
$$\mathcal{L}(H(t-k)f(t-k)) = e^{-ks}\mathcal{L}(f(t)).$$

Applying inverse Laplace transformation with these formulas in mind we get

$$\begin{aligned} x(t) &= 5(e^{-(t-2)} - e^{-2(t-2)})H(t-2) + 4(2e^{-2t} - e^{-t}) + 12(e^{-t} - e^{-2t}) \\ &= 5(e^{-(t-2)} - e^{-2(t-2)})H(t-2) - 4e^{-2t} + 8e^{-t}. \end{aligned}$$

2.9 Unsorted solved problems

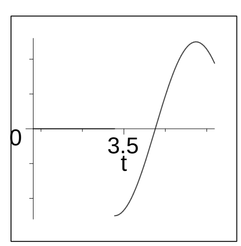


Problem:-1 Find $\mathcal{L}(g(t))$, where

$$g(t) = \begin{cases} 0 & \text{for } 0 < t < 1 \\ 3t & \text{for } 1 \le t \end{cases}$$

Solution: First recall that $\mathcal{L}(H(t-a)f(t-a)) = e^{-as}F(s)$. We therefore write $3t$ in shifted form: $3t = 3(t-1)+3$. Let $f(t) = 3t+3$. Then $f(t-1) = 3t$ and $H(t-1)f(t-1) = g(t)$, for $t > 0$. Hence

$$\begin{aligned} \mathcal{L}(g(t)) &= \mathcal{L}(H(t-1)f(t-1)) = e^{-s}\mathcal{L}(f(t)) \\ &= e^{-s}\mathcal{L}(3t+3) \\ &= e^{-s}\left(\frac{3}{s^2} + \frac{3}{s}\right). \end{aligned}$$



Problem:-2 Find $\mathcal{L}(g(t))$, where

$$g(t) = \begin{cases} 0 & \text{for } 0 < t < \pi \\ \cos t & \text{for } \pi \le t \end{cases}$$

Solution: Observe that $\cos t = -\cos(t-\pi)$. Setting $f(t) = -\cos t$, we note that $g(t) = H(t-\pi)f(t-\pi)$. So

$$\begin{aligned}\mathcal{L}(g(t)) &= \mathcal{L}(H(t-\pi)f(t-\pi) \\ &= e^{-\pi s}\mathcal{L}(f(t)) \\ &= e^{-\pi s}\frac{-s}{s^2+1}.\end{aligned}$$

Problem:-3 Find the inverse Laplace transform of $F(s) = \dfrac{1}{s^3 - s^2 + s - 1}$.
Solution:

$$\begin{aligned}F(s) &= \frac{1}{(s-1)(s^2+1)} \\ &= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \frac{1}{2}\left(\mathcal{L}^{-1}(\frac{1}{s-1}) - \mathcal{L}^{-1}(\frac{s}{s^2+1}) - \mathcal{L}^{-1}(\frac{1}{s^2+1})\right) \\ &= \frac{1}{2}\left(e^t - \cos t - \sin t\right).\end{aligned}$$

Problem:-4 Find the inverse Laplace transform of $F(s) = \dfrac{2}{s} + \dfrac{e^{-3s}}{s^2}$.
Solution:

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}(\frac{2}{s}) + \mathcal{L}^{-1}(\frac{e^{-3s}}{s^2}) \\ &= 2 + H(t-3)(t-3).\end{aligned}$$

Problem:-5 Solve $f''(t) + f(t) = t$, where $f(0) = 1,\ f'(0) = -2$.
Solution:

$$\begin{aligned}\mathcal{L}(f''(t)) + \mathcal{L}(f(t)) &= \mathcal{L}(t) \\ \left(s^2F(s) - sf(0) - f'(0)\right) + F(s) &= \frac{1}{s^2}\end{aligned}$$

$$s^2F(s) - s + 2 + F(s) = \frac{1}{s^2}.$$

Solving for $F(s)$;

$$F(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1},$$

and applying inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right) = t + \cos t - 3\sin t = f(t).$$

Problem:-6 Solve the initial value problem $y''(t) + y(t) = f(t)$, $y(0) = y'(0) = 0$, where $f(t) = n+1$ for $n\pi \leq t < (n+1)\pi$, $n \in \mathbb{N}$, i.e. $f(t) = \sum_{k=0}^{\infty} H(t - k\pi)$.

Solution: We plan to take the Laplace transform of both sides of the differential equation. For this observe that

$$\begin{aligned}
\mathcal{L}(y''(t)) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) \\
\mathcal{L}(y(t)) &= Y(s) \\
\mathcal{L}(f(t)) &= \mathcal{L}\left(\sum_{k=0}^{\infty} H(t-k\pi)\right) \\
&= \sum_{k=0}^{\infty} \mathcal{L}(H(t-k\pi)) = \sum_{k=0}^{\infty} \frac{e^{-k\pi s}}{s}.
\end{aligned}$$

Putting these together, the differential equation becomes

$$s^2Y(s) + Y(s) = \frac{1}{s}\sum_{k=0}^{\infty} e^{-k\pi s}$$

and solving for $Y(s)$

$$\begin{aligned}
Y(s) &= \left(\frac{1}{s(s^2+1)}\right)\sum_{k=0}^{\infty} e^{-k\pi s} \\
&= \frac{1}{s}\sum_{k=0}^{\infty} e^{-k\pi s} - \frac{s}{s^2+1}\sum_{k=0}^{\infty} e^{-k\pi s}.
\end{aligned}$$

Before applying the inverse Laplace transform to both sides recall that $\mathcal{L}(H(t-a)\cos(t-a)) = (\frac{s}{s^2+1})e^{-as}$.

Define a new function

$$g(t) = \sum_{k=0}^{\infty} H(t - k\pi)\cos(t - k\pi).$$

We can finally apply the inverse Laplace transform to $Y(s)$ to find

$$\begin{aligned} \mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(\frac{1}{s}\sum_{k=0}^{\infty} e^{-k\pi s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\sum_{k=0}^{\infty} e^{-k\pi s}\right) \\ y(t) &= f(t) - g(t). \end{aligned}$$

Problem:-7 Solve the initial value problem $y''(t) + y(t) = 3\sin 2t$, $t \in [0, \infty]$, $y(0) = 1$, $y'(0) = -2$.

Solution: Letting $Y(s) = \mathcal{L}(y(t))$, note that

$$\begin{aligned} \mathcal{L}(y''(t)) &= s^2 Y(s) - sy(0) - y'(0) \\ &= s^2 Y(s) - s + 2, \\ \mathcal{L}(\sin 2t) &= \frac{2}{s^2+4}. \end{aligned}$$

Taking the inverse Laplace of both sides of the differential equation we get

$$s^2 Y(s) - s + 2 + Y(s) = \frac{6}{s^2+4}.$$

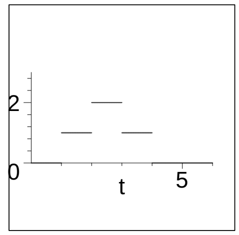
Solving for $Y(s)$ we get

$$Y(s) = \frac{s}{s^2+1} - \frac{2}{s^2+4}.$$

Taking the inverse Laplace transform gives

$$y(t) = \cos t - \sin 2t.$$

Problem:-8 Define a function $f(t)$ as



$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } 1 \leq t < 2 \\ 2 & \text{if } 2 \leq t < 3 \\ 1 & \text{if } 3 \leq t < 4 \\ 0 & \text{if } 4 \leq t \end{cases}$$

Note that $f(t) = H(t-1) + H(t-2) - H(t-3) - H(t-4)$.
Solve the initial value problem $y'' - 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$.

Solution: Taking the Laplace transform of both sides gives

$$s^2Y - 3sY + 2Y = \frac{1}{s}(e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}).$$

Set $A = (e^{-s} + e^{-2s} - e^{-3s} - e^{-4s})$. Then solving for Y gives

$$\begin{aligned} Y &= \frac{A}{s(s-1)(s-2)} \\ &= \frac{1}{2}\frac{A}{s} - \frac{A}{s-1} + \frac{1}{2}\frac{A}{s-2}. \end{aligned}$$

Recall that

$$\mathcal{L}(H(t-a)e^{b(t-a)}) = \frac{e^{as}}{s-b}.$$

Taking the inverse Laplace transform of Y gives

$$\begin{aligned} y(t) &= \sum_{k=1}^{2} \left(\frac{1}{2} - e^{t-k} + \frac{1}{2}e^{2(t-k)} \right) H(t-k) \\ &\quad - \sum_{k=3}^{4} \left(\frac{1}{2} - e^{t-k} + \frac{1}{2}e^{2(t-k)} \right) H(t-k) \end{aligned}$$

Problem:-9 Find the solution of the system

$$\frac{dx}{dt} - 6x + 3y = 8e^t$$

$$\frac{dy}{dt} - 2x - y = 4e^t$$

with initial conditions $x(0) = -1$, $y(0) = 0$.

Solution: Taking the Laplace transform of the system and simplifying we find

$$\begin{aligned} (s-6)X + 3Y &= \frac{-s+9}{s-1} \\ -2X + (s-1)Y &= \frac{4}{s-1} \end{aligned}$$

Solving for X and Y we find

$$\begin{aligned} X &= \frac{-s+7}{(s-1)(s-4)} = \frac{-2}{s-1} + \frac{1}{s-4} \\ Y &= \frac{2}{(s-1)(s-4)} = \frac{-2/3}{s-1} + \frac{2/3}{s-4}. \end{aligned}$$

Applying inverse Laplace transform to these equations gives

$$\begin{aligned} x(t) &= -2e^t + e^{4t} \\ y(t) &= -\frac{2}{3}e^t + \frac{2}{3}e^{4t}. \end{aligned}$$

Problem:-10 Find that solution of
$u_x(x,t) = 2u_t(x,t) + u(x,t)$, $u(x,0) = 6e^{-3x}$, which is bounded for $x > 0$, $t > 0$.

Solution: First note that

$$\begin{aligned}\mathcal{L}(u_x(x,t)) &= \int_0^\infty e^{-st}\frac{\partial u(x,t)}{\partial x}dt \\ &= \frac{d}{dx}\int_0^\infty e^{-st}u(x,t)dt \\ &= \frac{d}{dx}U(x,s).\end{aligned}$$

It follows from general properties of Laplace transform that

$$\mathcal{L}(u_t(x,t)) = sU(x,s) - u(x,0).$$

Putting these together, the given PDE transforms to

$$\frac{d}{dx}U - (2s+1)U = -12e^{-3x}.$$

Multiplying both sides by the integration factor $e^{-(2s+1)x}$ gives

$$\frac{d}{dx}(Ue^{-(2s+1)x}) = -12e^{-(2s+4)x}.$$

Integrating this gives

$$Ue^{-(2s+1)x} = \frac{6}{s+2}e^{-(2s+4)x} + c,$$

or

$$U = \frac{6}{s+2}e^{-3x} + ce^{(2s+1)x}.$$

Since $u(x,t)$ must stay bounded as $x \to \infty$, likewise $U(x,s)$ must stay bounded when $x \to \infty$. So we must choose $c = 0$, and then we have

$$U(x,s) = \frac{6}{s+2}e^{-3x},$$

and hence

$$u(x,t) = 6e^{-2t-3x}.$$

2.10 Unsorted Exercises

These exercises are taken from [5, 7, 8].

Exercise:-1 Find $\mathcal{L}(5t-2)$.

Ans: $\dfrac{5}{s^2} - \dfrac{2}{s}$.

Exercise:-2 Find $\mathcal{L}(t^3 + 8e^{-t} + 1)$.

Ans: $\dfrac{6}{s^4} + \dfrac{8}{s+1} + \dfrac{1}{s}$.

Exercise:-3 Find $\mathcal{L}(a\sin(at) + b\sin(bt))$.

Ans: $\dfrac{a^2}{s^2+a^2} + \dfrac{b^2}{s^2+b^2}$.

Exercise:-4 Find $\mathcal{L}(\cos(at-\alpha))$.

Ans: $\dfrac{s\cos\alpha + a\sin\alpha}{s^2+a^2}$.

Exercise:-5 Find $\mathcal{L}^{-1}(\dfrac{1}{s^4})$.

Ans: $t^3/6$.

Exercise:-6 Find $\mathcal{L}^{-1}(\dfrac{s+1}{s^3})$.

Ans: $t + t^2/2$.

Exercise:-7 Find $\mathcal{L}^{-1}(\dfrac{2s-5}{s^2+9})$.

Ans: $2\cos(3t) - (5/3)\sin(3t)$.

Exercise:-8 Find $\mathcal{L}^{-1}(\dfrac{7!}{(s-3)^8})$.

Ans: t^7e^{3t}.

Exercise:-9 Solve $y'' + 5y' + 6y = 3$, with $y(0) = 2$, $y'(0) = 0$.
Ans: $y = (1/2) + (9/2)e^{-2t} - 3e^{-3t}$.

Exercise:-10 Solve $y'' + 2y' + y = \sin t$, with $y(0) = 3$, $y'(0) = 1$.
Ans: $y = (9/2)te^{-t} + (7/2)e^{-t} - (1/2)\cos t$.

Exercise:-11 ([5, p50]) Solve the differential equation
$\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = 5\delta(t-2)$, with $x(0) = 4$, $x'(0) = 0$.

Ans: $5(e^{-(t-2)} - e^{-2(t-2)})H(t-2) + 8e^{-t} - 4e^{-2t}$.

Exercise:-12 [7, p456] Solve the following linear system using Laplace transform technique:

$$\begin{aligned} \frac{dx}{dt} + y &= 3e^{2t} \\ \frac{dy}{dt} + x &= 0 \end{aligned}$$

$$x(0) = 2 \qquad y(0) = 0.$$

Ans: $x = -\dfrac{e^t}{2} + \dfrac{e^{-t}}{2} + 2e^{2t},\ y = \dfrac{e^t}{2} + \dfrac{e^{-t}}{2} - e^{2t}$.

Exercise:-13 [7, p457] Solve the following linear system using Laplace transform technique:

$$\begin{aligned} 2\frac{dx}{dt} + \frac{dy}{dt} - x - y &= e^{-t} \\ \frac{dx}{dt} + \frac{dy}{dt} + 2x + y &= e^{t} \end{aligned}$$

$$x(0) = 2 \qquad y(0) = 1.$$

Ans: $x = 8\sin t + 2\cos t,\ y = -13\sin t + \cos t + \dfrac{e^t}{2} - \dfrac{e^{-t}}{2}$.

Exercise:-14 [7, p457] Solve the following linear system using Laplace transform technique:

$$\begin{aligned}\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + \frac{dy}{dt} + 2x - y &= 0\\ \frac{dx}{dt} + \frac{dy}{dt} - 2x + y &= 0\end{aligned}$$

$$x(0) = 0 \quad x'(0) = 0 \qquad y(0) = -1.$$

Ans: $x = -1 + 2e^t - e^{2t},\ y = -2 + e^t.$

Exercise:-15 [8, p484] Solve the following differential equation using Laplace transform technique:

$f''(t) - f'(t) - 2f(t) = e^{-t}\sin 2t$, with $f(0) = 0$ and $f'(0) = 2$.

Ans: $f(t) = \dfrac{28}{39}e^{2t} - \dfrac{5}{6}e^{-t} - \dfrac{1}{13}e^{-t}\sin 2t + \dfrac{3}{26}e^{-t}\cos 2t.$

3 The z-transform

Suppose $f(t)$ is a continuous function and we sample this function at time intervals of T, thus obtaining the data

$$f(0), f(T), f(2T), \ldots, f(nT), \ldots$$

Recall that the impulse function at $t = T$ is denoted by $\delta(t-T)$. If we denote by $f^*(t)$ the sampled function we can write

$$\begin{aligned} f^*(t) &= f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \cdots \\ &= \sum_{n=0}^{\infty} f(nT)\delta(t-nT) \end{aligned}$$

The Laplace transform of this function then becomes

$$\begin{aligned} F^*(s) &= \mathcal{L}(f^*(t)) \\ &= \sum_{n=0}^{\infty} f(nT)\mathcal{L}(\delta(t-nT)) \\ &= \sum_{n=0}^{\infty} f(nT)e^{-nTs} \end{aligned}$$

If we now set

$$z = e^{sT} \text{ or equivalently } s = \frac{1}{T}\log(z)$$

then we can define

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

This function $F(z)$ is called the **z-transform** of the discrete time signal function $f(nT)$,

$$F(z) = \mathcal{Z}(f(t)).$$

In other words

$$\begin{aligned} \mathcal{Z}(f(t)) &= F(z) \\ &= F^*(s) = F^*(\frac{1}{T}\log(z)) \\ &= \left[\mathcal{L}\left(\sum_{n=0}^{\infty} f(nT)(\delta(n-nT))\right)\right]_{s=\frac{1}{T}\log(z)}. \end{aligned}$$

Sometimes, as a suggestive notation, we write $\mathcal{Z}(f(nT))$ instead of $\mathcal{Z}(f(t))$.

Example:-3.8 Find $\mathcal{Z}(H(nT))$. Here we are sampling the function $f(t) = H(t)$, the unit step function, or the Heaviside function, and obtaining the sample $f(n) = 1$ for all $n \geq 0$.

Solution:

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} 1\, z^{-n} = 1 + z^{-1} + z^{-2} + \cdots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}. \end{aligned}$$

Hence we find that

$$\mathcal{Z}(H(nT)) = \frac{z}{z-1} \text{ for } |z| > 1.$$

Example:-3.9 Find the z-transform of the sampled function $f(nT)$ for $f(t) = t$, (the ramp function).

Solution: We find that $f(nT) = nT$. Hence

$$\begin{aligned} F(z) &= Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots \\ &= \frac{T}{z(1-z^{-1})^2} \\ &= \frac{Tz}{(z-1)^2}. \end{aligned}$$

Hence

$$\mathcal{Z}(nT) = \frac{Tz}{(z-1)^2}, \text{ for } |z| > 1.$$

3.1 Elementary properties of z-transform

In this section we list some elementary properties of z-transform which follow from the basic definitions. Here $\alpha, \beta \in \mathbb{C}$ and $n, m \in \mathbb{N}$.

- $\mathcal{Z}(\alpha f_1(n) \pm \beta f_2(n)) = \alpha\mathcal{Z}(f_1(n)) \pm \beta\mathcal{Z}(f_2(n))$.

- $\mathcal{Z}(f(n-m)H(n-m)) = z^{-m}\mathcal{Z}(f(n))$.

- $\mathcal{Z}(f(n+m)) = z^m \left(\mathcal{Z}(f(n)) - \sum_{k=0}^{m-1} f(k)z^{m-k}\right)$. In particular
$\mathcal{Z}(f(n+1)) = zF(z) - zf(0)$,
$\mathcal{Z}(f(n+2)) = z^2F(z) - z^2f(0) - zf(1)$, and
$\mathcal{Z}(f(n+3)) = z^3F(z) - z^3f(0) - z^2f(1) - zf(2)$.

- $\lim_{t\to 0} f^*(t) = \lim_{z\to\infty} F(z)$.

- $\lim_{t\to\infty} f^*(t) = \lim_{z\to 1} \frac{z-1}{z}F(z)$.

- If $f(n) = f(n-N)$, i.e. the sampled data is periodic with period N, then

$$\mathcal{Z}(f(n)) = \frac{\sum_{k=0}^{N-1} f(k)z^{-k}}{1-z^{-N}}.$$

3.2 A table of z-transforms

In the following list we describe $F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$, where $f(n)$ is the given function. Here again $\alpha \in \mathbb{C}$ and $n, m \in \mathbb{N}$.

- If $f(m) = \alpha$ and $f(n) = 0$ for $n \neq m$, then $F(z) = \alpha z^{-m}$.

- $\mathcal{Z}(1) = \dfrac{z}{z-1}$.

- $\mathcal{Z}(n) = \dfrac{z}{(z-1)^2}$.

- $\mathcal{Z}(n^2) = \dfrac{z(z+1)}{(z-1)^3}$.
- $\mathcal{Z}(e^{\alpha n}) = \dfrac{z}{z-e^{\alpha}}$.
- $\mathcal{Z}(ne^{\alpha n}) = \dfrac{ze^{\alpha}}{(z-e^{\alpha})^2}$.
- $\mathcal{Z}(\sin \alpha n) = \dfrac{z \sin \alpha}{z^2+1-2z\cos\alpha}$.
- $\mathcal{Z}(\cos \alpha n) = \dfrac{z(z-\cos\alpha)}{z^2+1-2z\cos\alpha}$.
- $\mathcal{Z}(\alpha^n) = \dfrac{z}{z-\alpha}$.
- $\mathcal{Z}(n\alpha^n) = \dfrac{\alpha z}{(z-\alpha)^2}$.
- $\mathcal{Z}(\dfrac{\alpha^n}{n!}) = e^{\alpha/z}$.
- $\mathcal{Z}(\alpha^n f(n)) = F(z/\alpha)$.
- $\mathcal{Z}(\sum_{k=0}^{n} f(k)) = \dfrac{zF(z)}{z-1}$.
- $\mathcal{Z}(\sum_{k=0}^{n} f_1(k) f_2(n-k)) = F_1(z)F_2(z)$.

3.3 The inverse z-transform

The z-transform of a given sequence is unique. To find the function $f(n)$ when $F(z)$ is given we can employ one of the following three methods:

Power series method
Using the description for $F(z)$ we try to write it in the form

$$F(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

Then

$$f(n) = a_n.$$

It is in general difficult to find a closed formula for the Laurent series expansion of $F(z)$, but when it is possible to do so this method works well.

Example:-3.10 If $F(z) = z/(z-\alpha)$, find $f(n)$.

Solution:

$$\begin{aligned} F(z) &= \frac{z}{z-\alpha} = \frac{1}{1-\alpha/z} \\ &= 1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \frac{\alpha^3}{z^3} + \cdots \\ &= \sum_{n=0}^{\infty} \alpha^n z^{-n}, \end{aligned}$$

and hence $f(n) = \alpha^n$.

Partial fractions method
This method works when $F(z)$ is a rational function of z. You convert $F(z)$ to a partial fraction form and then recognize the parts from a z-transform table.

Observe that most forms of rational $F(z)$ has the same degree in the numerator as the denominator. In such cases you should start with $F(z)/z$, obtain its partial fraction form, and multiply both sides by z to obtain the required form for $F(z)$.

Example:-3.11 Find $f(n)$ when $F(z) = \dfrac{z^2}{(z+1)(z-2)}$.

Solution:

$$\begin{aligned} \frac{F(z)}{z} &= \frac{z}{(z+1)(z-2)} \\ &= \frac{1}{3}\frac{1}{z+1} + \frac{2}{3}\frac{1}{z-2}. \\ F(z) &= \frac{1}{3}\frac{z}{z+1} + \frac{2}{3}\frac{z}{z-2} \\ &= \frac{1}{3}\mathcal{Z}((-1)^n) + \frac{2}{3}\mathcal{Z}(2^n) \\ &= \mathcal{Z}(\frac{(-1)^n + 2^{n+1}}{3}), \quad \text{and} \\ f(n) &= \frac{(-1)^n + 2^{n+1}}{3}. \end{aligned}$$

The residue method
This method is summarized in your text book, [2, Exercise 9, page 157]. As a result it can be shown that if $F(z)$ is the z-transform of $f(n)$, then

$$f(n) = \frac{1}{2\pi i} \int_C z^{n-1} F(z) dz$$

where C is a closed contour including the disk $|z| \leq R$ in its interior, where $|z| > R$ is the region of convergence, or the region of analyticity, for the function $F(z)$. This integral is then evaluated using residue theory. i.e.

$$f(n) = \sum \text{Res}(z^{n-1}F(z)).$$

Example:-3.12 Find $f(n)$ if its z-transform is $F(z) = 4z/(3z^2 - 2z - 1)$.

Solution: $\underset{\text{z=1}}{\text{Res}}(z^{n-1}F(z)) = 1$, $\underset{\text{z=-1/3}}{\text{Res}}(z^{n-1}F(z)) = (-1/3)^{n-1}$. Sum of the residues is $1 + (-1/3)^{n-1}$, which is the expression for $f(n)$.

3.4 Solving difference equations

A difference equations is an equation of the form

$$a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k) = g(n,k)$$

where the a_i's are constants, $g(n,k)$ is a given function, and we try to find f. These equations are also known as recurrence equations. Note that in the above set up you must specify $f(0)$, ..., $f(k-1)$ to find f.

To solve such an equation using z-transform, you take the z-transform of both sides of the equation to obtain an algebraic equation in $F(z)$. You solve for $F(z)$ from this equation and take the inverse z-transform to find f.

Example:-3.13 Find a closed form expression for the general term of the Fibonacci sequence which is defined by $F_1 = F_2 = 1$ and $F_n + F_{n+1} = F_{n+2}$ for $n \geq 1$.

Solution: We define $f(n) = F_{n+1}$ for $n \geq 0$. Then recurrence equation becomes $f(n) + f(n+1) = f(n+2)$ with $f(0) = f(1) = 1$. Using the list of elementary z-transforms we find that transforming both sides of this equation gives

$$F(z) + (zF(z) - z) = z^2 F(z) - z^2 - z.$$

Solving this for $F(z)$ we find

$$F(z) = \frac{z^2}{z^2 - z - 1} = \left(\frac{\phi}{\phi + 1/\phi}\right)\frac{z}{z - \phi} + \left(\frac{1/\phi}{\phi + 1/\phi}\right)\frac{z}{z + 1/\phi}$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Ratio. Applying inverse z-transform to $F(z)$ we find

$$\begin{aligned} f(n) &= \left(\frac{\phi}{\phi + 1/\phi}\right)(\phi)^n + \left(\frac{1/\phi}{\phi + 1/\phi}\right)(-\frac{1}{\phi})^n \\ &= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - (-\frac{1}{\phi})^{n+1}\right). \end{aligned}$$

Since $f(n) = F_{n+1}$, we obtain the following closed form formula for the general term of the Fibonacci sequence:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right), \text{ for } n > 2.$$

Example:-3.14 [5, Ex-5 p94] Solve the following difference equation:
$f(n+2) - 4f(n+1) + 4f(n) = 2^n$, with $f(0) = 1$, $f(1) = -1$.

Solution: Apply z-transform to both sides of this equation. Note first that:

$$\begin{aligned}
\mathcal{Z}(f(n)) &= F(z), \\
\mathcal{Z}(f(n+1)) &= zF(z) - zf(0) = zF(z) - z, \\
\mathcal{Z}(f(n+2)) &= z^2F(z) - z^2f(0) - zf(1) = z^2F(z) - z^2 + z, \\
\mathcal{Z}(2^n) &= \frac{z}{z-2}.
\end{aligned}$$

The difference equation then becomes

$$(z-2)^2F(z) - (z^2 - 5z) = \frac{z}{z-2}$$

and solving for $F(z)$ we find

$$F(z) = \frac{z^3 - 7z^2 + 11z}{(z-2)^3}.$$

The residue method to find the inverse z-transform of this function says that

$$f(n) = \operatorname*{Res}_{z=2} z^{n-1}F(z).$$

This residue is equal to $\dfrac{\phi''(2)}{2}$ where $\phi(z) = z^{n-1}(z^3 - 7z^2 + 11z)$.

Taking successive derivatives gives

$$\begin{aligned}
\phi(z) &= z^{n+2} - 7z^{n+1} + 11z^n, \\
\phi'(z) &= (n+2)z^{n+1} - 7(n+1)z^n + 11nz^{n-1}, \\
\phi''(z) &= (n+2)(n+1)z^n - 7(n+1)nz^{n-1} + 11n(n-1)z^{n-2}, \\
&= z^{n-2}\big((n+2)(n+1)z^2 - 7(n+1)nz + 11n(n-1)\big)
\end{aligned}$$

and putting in $z = 2$ gives

$$\begin{aligned}
\phi''(2) &= 2^{n-2}(n^2 - 13n + 8). \text{ Hence} \\
f(n) &= \frac{\phi''(2)}{2} = 2^{n-3}(n^2 - 13n + 8).
\end{aligned}$$

Example:-3.15 [4, Ex-5 p371] Solve the following difference equation:
$i_{n+2} - i_{n+1} + i_n = 0$, where $i_1 = 3i_0 - V/R$ and i_0, V and R are constants.

Solution: Let $I(z)$ denote the z-transform of i_n.

$$\begin{aligned}
\mathcal{Z}(i_n) &= I(z), \\
\mathcal{Z}(i_{n+1}) &= zI(z) - zi_0, \\
\mathcal{Z}(i_{n+2}) &= z^2 I(z) - z^2 i_0 - zi_1, \\
&= z^2 I(z) - z^2 i_0 - z(i_0 - V/R).
\end{aligned}$$

The difference equation becomes

$$z^2 I(z) - z^2 i_0 - z(3i_0 - V/R) - 4zI(z) + 4zi_0 + I(z) = 0$$

from which we find

$$I(z) = \frac{i_0 z^2 - \left(i_0 + \frac{V}{R}\right) z}{z^2 - 4z + 1}.$$

The residue method to invert this is easier than the other methods. The function $I(z)$ has two simple poles at

$$\begin{aligned}
z_1 &= 2 - \sqrt{3} \quad \text{and} \\
z_2 &= 2 + \sqrt{3}.
\end{aligned}$$

An easy calculation gives

$$\begin{aligned}
\operatorname*{Res}_{z\,=\,z_1} z^{n-1} I(z) &= \frac{i_0 + z_1 - \left(i_0 + \frac{V}{R}\right)}{-2\sqrt{3}} z_1^n, \quad \text{and} \\
\operatorname*{Res}_{z\,=\,z_2} z^{n-1} I(z) &= \frac{i_0 + z_2 - \left(i_0 + \frac{V}{R}\right)}{2\sqrt{3}} z_2^n,
\end{aligned}$$

Hence we get

$$\begin{aligned}
i_n &= \operatorname*{Res}_{z\,=\,z_1} z^{n-1} I(z) + \operatorname*{Res}_{z\,=\,z_2} z^{n-1} I(z), \quad n \geq 1 \\
&= \frac{i_0 + z_1 - \left(i_0 + \frac{V}{R}\right)}{-2\sqrt{3}} z_1^n + \frac{i_0 + z_2 - \left(i_0 + \frac{V}{R}\right)}{2\sqrt{3}} z_2^n, \\
&= \frac{i_0}{2\sqrt{3}} \left((z_2^{n+1} - z_1^{n+1}) + (z_1^n - z_2^n) \right) + \frac{V}{R} \frac{1}{2\sqrt{3}} (z_1^n - z_2^n) \\
&= i_0 \left(\sum_{k=0}^{\|n/2\|} \binom{n+1}{2k+1} 3^k 2^{n-2k} - \sum_{k=0}^{\|(n-1)/2\|} \binom{n}{2k+1} 3^k 2^{n-2k-1} \right) \\
&\quad - \frac{V}{R} \left(\sum_{k=0}^{\|(n-1)/2\|} \binom{n}{2k+1} 3^k 2^{n-2k-1} \right),
\end{aligned}$$

where $\|m\|$ stands for the greatest integer which is less than or equal to m.

The first few values of i_n are as follows:

$i_1 = 3\, i_0 - \frac{V}{R}$,	$i_2 = 11\, i_0 - 4\,\frac{V}{R}$,
$i_3 = 41\, i_0 - 15\,\frac{V}{R}$,	$i_4 = 153\, i_0 - 56\,\frac{V}{R}$,
$i_5 = 571\, i_0 - 209\,\frac{V}{R}$,	$i_6 = 2131\, i_0 - 780\,\frac{V}{R}$,
$i_{10} = 413403\, i_0 - 151316\,\frac{V}{R}$,	$i_{20} = 216695104121\, i_0 - 79315912984\,\frac{V}{R}$

Example:-3.16 Suppose you deposit m millions of TL to a bank savings account each month. The bank gives you $100c$ per cent interest per month, where $0 < c < 1$. Find how much money you will have at the end of the n-th month.

Solution: Let $f(n)$ denote the amount of money you will have at the end of the n-th month. You start with $f(0) = m$, which means that you first deposit m millions of TL, so have m millions TL to begin with. At the end of the first month you earn $(1+c)m$ millions of TL and deposit m millions TL more yourself, so at the end of the first month you have $f(1) = m\,(1 + (1 + c))$ millions TL at the bank.

Arguing similarly we see that the recursive relation that we have to solve is

$$f(n+1) = (1+c)f(n) + m, \quad \text{with} \quad f(0) = m.$$

Since this is an easy problem we will demonstrate the implementation of four different methods in solving it.

Induction Method: Use induction to show that

$$f(n) = \left((1+c)^{n+1} - 1\right)\frac{m}{c}, \quad \text{for} \quad n = 0, 1, 2, ...$$

The next three methods involve the z-transform technique. Take the z-transform of the given recursion equation, solve for $F(z)$ and find the inverse z-transform of the solution. As usual we have

$$\begin{aligned} \mathcal{Z}(f(n)) &= F(z), \\ \mathcal{Z}(f(n+1)) &= zF(z) - zf(0) \\ &= zF(z) - zm, \\ \mathcal{Z}(m) &= \frac{mz}{z-1}, \end{aligned}$$

and the recursion equation becomes

$$zF(z) - zm = (1+c)F(z) + \frac{mz}{z-1}.$$

Solving this for $F(z)$ gives

$$F(z) = \frac{z^2}{(z-1)(z-(1+c))}m.$$

Now we will demonstrate the use of the three methods of inversion on this function.

Power Series Method:

$$\begin{aligned} F(z) &= \frac{z^2}{(z-1)(z-(1+c))}m \\ &= \frac{1}{(1-1/z)(1-(1+c)/z)}m \\ &= \left(\sum_{n=0}^{\infty}\frac{1}{z^n}\right)\left(\sum_{n=0}^{\infty}\frac{(1+c)^n}{z^n}\right)m \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(1+c)^k\right)\frac{m}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{[(1+c)^{n+1}-1]m}{c}\frac{1}{z^n}, \end{aligned}$$

and hence the coefficient of $1/z^n$ gives the required function $f(n)$.

Partial Fractions Method:

$$\begin{aligned}
F(z) &= \left[\frac{z}{(z-1)(z-(1+c))}\right] zm \\
&= \left[-\frac{1}{c}\frac{1}{z-1} + \frac{1+c}{c}\frac{1}{z-(1+c))}\right] zm \\
&= \left[-\frac{1}{c}\frac{z}{z-1} + \frac{1+c}{c}\frac{z}{z-(1+c))}\right] m \\
&= -\frac{m}{c}\mathcal{Z}(1) + \frac{(1+c)m}{c}\mathcal{Z}((1+c)^n) \\
&= \mathcal{Z}(\frac{[(1+c)^{n+1}-1]m}{c}). \\
f(n) &= \frac{[(1+c)^{n+1}-1]m}{c}.
\end{aligned}$$

Residue Method: We note that $z^{n-1}F(z) = \dfrac{z^{n+1}m}{(z-1)(z-(1+c))}$. Calculating its residues we find

$$\begin{aligned}
\operatorname*{Res}_{z\,=\,1}\left(z^{n-1}F(z)\right) &= -\frac{m}{c}, \\
\operatorname*{Res}_{z\,=\,1+c}\left(z^{n-1}F(z)\right) &= \frac{(1+c)^{n+1}m}{c}.
\end{aligned}$$

Finally, adding up the residues we find the expected formula

$$f(n) = \frac{[(1+c)^{n+1}-1]m}{c}.$$

3.5 Unsorted exercises

These exercises are mostly taken from [4, 5, 8].
Determine the z-transform of the following samples:

Exercise:-1 $\cosh \alpha n$. Ans: $\dfrac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$.

Exercise:-2 $\sinh \alpha n$. Ans: $\dfrac{\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$.

Determine the inverse z-transform of the following functions:

Exercise:-3 $\dfrac{z}{(z-3)^2}$. Ans: $n(3^{n-1})$.

Exercise:-4 $\dfrac{z}{z^2+1}$. Ans: $\sin\dfrac{n\pi}{2}$.

Exercise:-5 $\dfrac{4z}{4z^2-2z\sqrt{3}+1}$. Ans: $\left(\dfrac{1}{2}\right)^{n-2}\sin\dfrac{n\pi}{6}$.

Exercise:-6 $\dfrac{2z^3}{(z-2)^3}$. Ans: $(n^2+3n+2)2^n$.

Exercise:-7 $z\left(e^{1/z}-1\right)$. Ans: $1/(n+1)!$.

Exercise:-8 $\sinh\dfrac{2}{z}$. Ans: $(1-(-1)^n)\dfrac{2^{n-1}}{n!}$.

Solve the following difference equations:

Exercise:-9 $f(n+1)+2f(n)=(-1)^n$, with $f(0)=-2$.
Ans: $f(n)=(-1)^n-3(-2)^n$.

Exercise:-10 $x(n+2)+5x(n+1)+6x(n)=3$, with $x(0)=-2$, $x(1)=1$.
Ans: $x(n)=(1/4)-6(-2)^n+(15/4)(-3)^n$.

Exercise:-11 $2f(n+3)-3f(n+2)+f(n)=0$, with $f(0)=0$, $f(1)=1$, $f(2)=-4$. Ans: $f(n)=-(8/3)(-1/2)^n+(8/3)-3n$.

Exercise:-12 $x(n+2)-2x(n+1)+x(n)=0$, with $x(0)=A$, $x(1)=B$.
Ans: $x(n)=A+(B-A)n$

Exercise:-13 $y(n+2)-\sqrt{3}y(n+1)+y(n)=0$, with $y(0)=1$, $y(1)=\sqrt{3}$.
Ans: $y(n)=\cos(n\pi/6)+\sqrt{3}\sin(n\pi/6)$.

Exercise:-14 $a(n+2)-5a(n+1)+6a(n)=1$, with $a(0)=2$, $a(1)=3$.
Ans: $a(n)=(1-3^n+2^{n+2})/2$.

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