Math 213 Advanced Calculus
Final Exam
SOLUTIONS

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1) Construct two sequences of real numbers, $x_n \geq 0$ and $y_n \geq 0$, such that $\limsup_{n\to\infty}(x_n y_n) < (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n)$

Several such constructions are possible. Take for example

$$x_n = \begin{cases} 1 + \frac{1}{n} & n \text{ odd} \\ \frac{1}{n} & n \text{ even} \end{cases}$$

and

$$y_n = \begin{cases} 1 + \frac{1}{n} & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases}$$

Then $x_n y_n = (1/n) + (1/n^2)$ and converges to zero. Hence $\limsup_{n\to\infty}(x_n y_n) = 0$, $\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = 1$.

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2) Let $f(x) = x\tan^2 x$ for $x \in (0, \pi/2)$. Calculate $(f^{-1})'(\pi)$.

See the Inverse Function Theorem on page 96. Using the formula $(f^{-1})'(y_0) = 1/f'(x_0)$, where $y_0 = f(x_0)$, and noting that here $x_0 = \pi/3$ one easily finds that the answer is $(3 + (8/\sqrt{3})\pi)^{-1}$.

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3) Show that I_p is conditionally convergent for $0 < p \leq 1$, and absolutely convergent for $p > 1$, where $I_p = \displaystyle\int_1^\infty \frac{\sin x}{x^p} dx$.

For the conditionally convergent part look at Example 5 on page 130-131. For the second part use the Comparison Theorem on page 129.

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4) Find the <u>interval</u> of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n+1)}{n!} x^n$.

Ratio test gives $|x| < 1/2$ for absolute convergence. When $|x| = 1/2$ the general term of the series does not converge to zero, so the series diverges.

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5) Show that in any metric space a compact set is always closed.

One proof is on page 236, Remark 4. This proof depends on Theorem 5.15 on page 235. Reading the two proofs carefully one sees that no property of $\mathbb{R}^n$ is used other than the existence of a metric, so the proof goes for an arbitrary metric space. I gave another proof in class which started by choosing a point y outside the compact proper set E and considering the open cover $\cup_{x \in E} B_{d(x,y)/2}(x)$. A finite subcover exists. Choosing $r > 0$ smaller than the radii of the balls in this finite subcover we see that the open ball $B_r(y)$ totally lies in the complement of E. Thus E^c is open and E is closed.
