Math 213 Advanced Calculus
Midterm Exam II
Solution Set

November 23, 2000 — Ali Sinan Sertöz

1) Construct a function $f : [1, \infty) \to \mathbb{R}$ which is unbounded on $[1, \infty)$ but is improperly integrable there in the sense that $\lim_{R\to\infty} \int_1^R f(x)dx$ exists and is finite.

Several such constructions are possible. One such function is given by

$$f(x) = \begin{cases} 2^n & \text{if } n < x \leq n + \frac{1}{2^{2n}} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f is unbounded. Notice however that $\int_n^{n+1} f(x)dx = 1/2^n$, so $\int_1^\infty f(x)dx = \lim_{N\to\infty} \sum_{n=1}^N \int_n^{n+1} f(x)dx = \lim_{n\to\infty} \sum_{n=1}^N \frac{1}{2^n} = 1$.

2) Assume that $f : [3, 5] \to \mathbb{R}$ is one-to-one. Assume further that f' exists and is integrable on $[3, 5]$, and that $f(3) = 7$, $f(5) = 8$. Calculate

$$\int_3^5 f(x)dx + \int_7^8 f^{-1}(x)dx.$$

Putting $x = f(t)$ we find that $\int_7^8 f^{-1}(x)dx = \int_3^5 f^{-1}(f(t))f'(t)dt = \int_3^5 tf'(t)dt$. Then $\int_3^5 f(x)dx + \int_7^8 f^{-1}(x)dx = \int_3^5 f(x)dx + \int_3^5 xf'(x)dx = \int_3^5 (xf(x))'dx = 5f(5) - 3f(3) = 19$.

3) Let $\{a_n\}$ be a sequence of real numbers such that for some real number $p > 1$ we have; $\lim_{n\to\infty} n^p a_n = A$, with $A \in \mathbb{R}$. Show that $\sum_{n=1}^\infty a_n$ converges.

Let $\epsilon > 0$ be chosen arbitrarily. Then there is an N such that for all $n \geq N$ we have

$$A - \frac{\epsilon}{2} < n^p a_n < A + \frac{\epsilon}{2}.$$

$$-\frac{\epsilon}{2}\frac{1}{n^p} < a_n - \frac{A}{n^p} < \frac{\epsilon}{2}\frac{1}{n^p} \tag{1}$$

Let $L_K = \sum_{n=N}^{K} \frac{1}{n^p}$, and $L = \lim_{K\to\infty} L_K$. Now sum up all sides of equation 1 from $n = N$ to K to obtain

$$-\frac{\epsilon}{2}L_K < \sum_{n=N}^{K} a_n - AL_K < \frac{\epsilon}{2}L_K$$

and since $0 < L_K < L$ we have

$$-\frac{\epsilon}{2}L < \sum_{n=N}^{K} a_n - AL_K < \frac{\epsilon}{2}L$$

$$-\frac{\epsilon}{2} < \frac{1}{L}\sum_{n=N}^{K} a_n - \frac{1}{L}AL_K < -\frac{\epsilon}{2}$$

$$|\frac{1}{L}\sum_{n=N}^{K} a_n - \frac{1}{L}AL_K| < \frac{\epsilon}{2}. \tag{2}$$

Since $\lim_{K\to\infty} \frac{A}{L}L_K = A$, there is an index K_0 such that for all $K \geq K_0$ we have

$$|\frac{A}{L}L_K - A| < \frac{\epsilon}{2}. \tag{3}$$

Now combining equations 2 and 3 we have for all $K \geq K_0$

$$|\frac{1}{L}\sum_{n=N}^{K} a_n - A| \leq |\frac{1}{L}\sum_{n=N}^{K} a_n - \frac{A}{L}L_K| + |\frac{A}{L}L_K - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which is equivalent to saying that $\sum_{n=N}^{\infty} a_n = AL$, and which in turn implies that the series $\sum_{n=1}^{\infty} a_n$ converges.

4) Show that $\cos(1)$ is irrational.

Assume $\cos(1) = \frac{m}{n}$ for some integers m and $n > 0$. Using the series expansion of cosine we have

$$\frac{m}{n} = 1 - \frac{1}{2!} + \cdots + (-1)^k \frac{1}{(2k)!} + \cdots$$

Assuming $2k \leq n < 2k+1$, multiply both sides by $(-1)^{k+1}n!$ to obtain

$$\begin{aligned}(-1)^{k+1}(n-1)!m &= [(-1)^{k+1}n!(1-\frac{1}{2!}+\cdots+(-1)^k\frac{1}{(2k)!})] \\ &\quad +\frac{n!}{(2k+2)!}-\frac{n!}{(2k+4)!}+\cdots\end{aligned}$$

Letting the expression inside the square brackets on the right hand side to be A and setting $B = (-1)^{k+1}(n-1)!m - A$, we see that B is an integer which satisfies

$$0 < \frac{n!}{(2k+2)!} - \frac{n!}{(2k+4)!} < B < \frac{n!}{(2k+2)!} < 1$$

which is a contradiction. Hence $\cos(1)$ is irrational.

5) Consider the set $A = \cup_{n=1}^{\infty}\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \geq 1+1/n\}$. Describe the interior, the closure and the boundary of A. Is A closed, open or neither?

$A = \{(x,y) \in \mathbb{R}^2 |\ x^2+y^2 > 1\ \}$. Therefore A is open and the interior of A is equal to A. The closure of A is $\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \geq 1\}$, and the boundary of A is the unit circle $x^2+y^2=1$.