Notes on Fourier Series
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Definitions: Throughout these notes f will be a function which is integrable and periodic on $\mathbb{R}$ with period 2π. We will mainly consider f on the domain $[-\pi, \pi]$.

The Fourier coefficients of f are the following numbers:

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\, dt,\ k = 0, 1, \ldots,$$

$$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\, dt,\ k = 1, 2, \ldots,$$

The Fourier series $(Sf)(x)$ associated to $f(x)$ is the series

$$(Sf)(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} a_k(f)\cos kx + b_k(f)\sin kx.$$

As usual, we employ partial sums to manage infinite series. The N-th Fourier sum $(S_N f)(x)$ of $f(x)$ is the Fourier polynomial defined by

$$(S_0 f)(x) = \frac{a_0(f)}{2},\ \text{and}\ (S_N f)(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{N} a_k(f)\cos kx + b_k(f)\sin kx,\ \text{where}\ N = 1, 2, \ldots.$$

These Fourier polynomials are used to define the N-th Cesàro sums $(\sigma_N f)(x)$ associated to $f(x)$ as follows:

$$(\sigma_N f)(x) = \frac{(S_0 f)(x) + \cdots + (S_N f)(x)}{N+1},\ \text{for}\ N = 0, 1, \ldots.$$

Notice that

$$\begin{aligned}(\sigma_N f)(x) &= \frac{a_0(f)}{2} + \frac{N}{N+1}\left[a_1(f)\cos x + b_1(f)\sin x\right] + \cdots \\ &\quad + \frac{N+1-k}{N+1}\left[a_k(f)\cos kx + b_k(f)\sin kx\right] + \cdots \\ &\quad + \frac{1}{N+1}\left[a_N(f)\cos Nx + b_N(f)\sin Nx\right],\end{aligned}$$

which follows from rearranging the terms. This is no problem for any particular N but when we decide to take the limit of $(\sigma_N f)(x)$ as N goes to infinity, such a rearrangement will not give the correct limit unless the Fourier series itself converges absolutely. In that case both the Fourier polynomials and the Cesàro polynomials converge to the same value, though at different rates.

Main questions: We are naturally interested in checking how well the trigonometric polynomials $(S_N f)(x)$ and $(\sigma_N f)(x)$ approximate the function $f(x)$. This can be pointwise or uniform. Hence we can formulate our questions as:
1) Does there exist a point $x \in [-\pi, \pi]$ such that $\lim_{N\to\infty}(S_N f)$ or $\lim_{N\to\infty}(\sigma_N f)(x))$ exist and is equal to $f(x)$?
2) Does $(S_N f)$ or $(\sigma_N f)$ converge to f uniformly on any interval $[a, b] \subset [-\pi, \pi]$?

First results: If no further properties of f are assumed, then we do not know anything about the relation of the above trigonometric series with f. In fact there are integrable functions whose Fourier series diverge everywhere. But with a very little assumption we can get convergence of Cesàro polynomials.

Fejér Theorem: If for some point x, the right and left limits $f(x+)$ and $f(x-)$ exist, then

$$\lim_{N\to\infty} (\sigma_N f)(x)) = \frac{f(x+) + f(x-)}{2}.$$

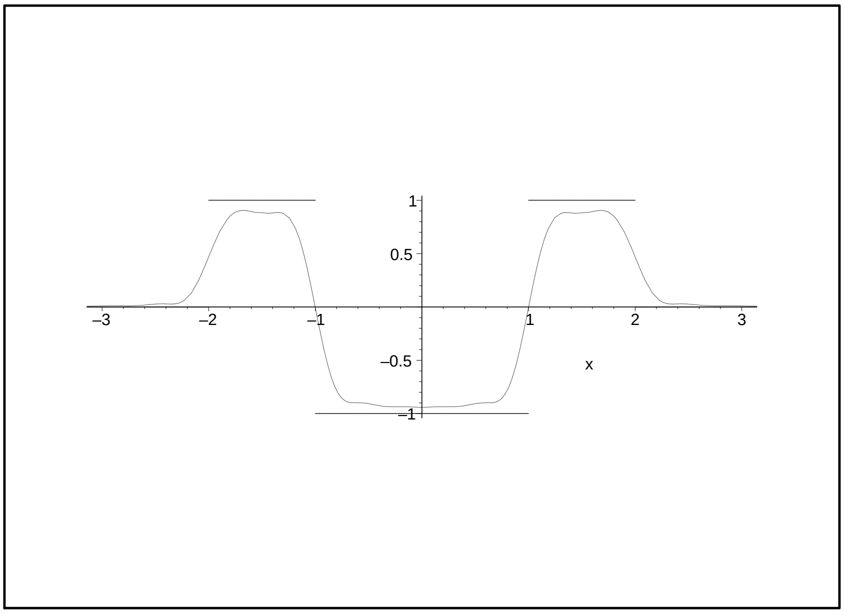
In particular, if f is continuous at x, then $\lim\limits_{N\to\infty} (\sigma_N f)(x)) = f(x)$. In fact, when f is continuous on (a, b), then $(\sigma_N f)$ converges to f uniformly as N goes to infinity on any closed subinterval of (a, b).

In the statement about uniform continuity we paid special attention to stay away from the end points. This is because there may be a discontinuity there in which case the convergence of Cesàro sum depends on the right and left limits. If f is continuous throughout $\mathbb{R}$, then the convergence of $(\sigma_N f)$ is uniform.

To show how nicely the Cesàro polynomials approximate a function we consider the following example.

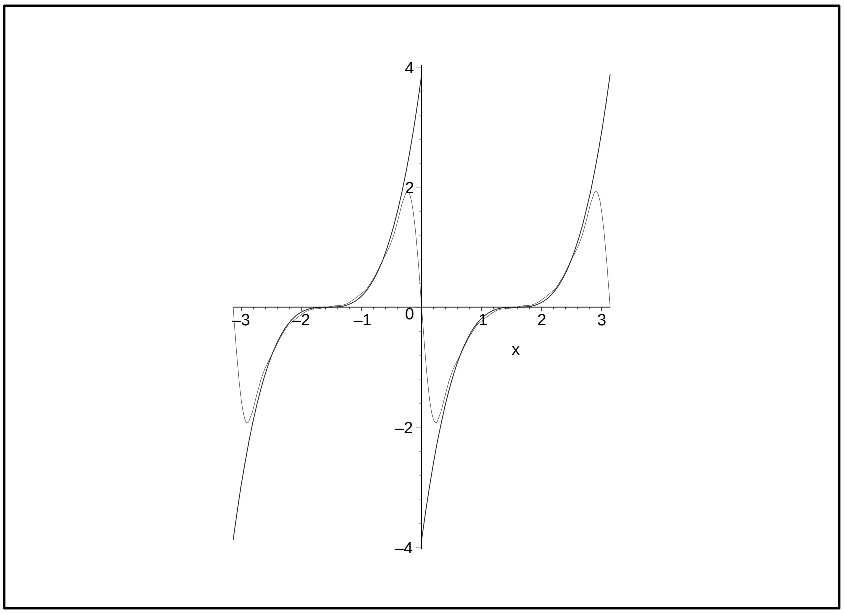
For $x \in [-\pi, \pi]$ define $f(x) = \begin{cases} 0, & x \in [-\pi, -2] \cup [2, \pi] \\ 1, & x \in [-2, -1] \cup [1, 2] \\ -1, & x \in [-1, 1]. \end{cases}$

The following graph plots $y = f(x)$ together with $y = (\sigma_{15} f)(x)$.

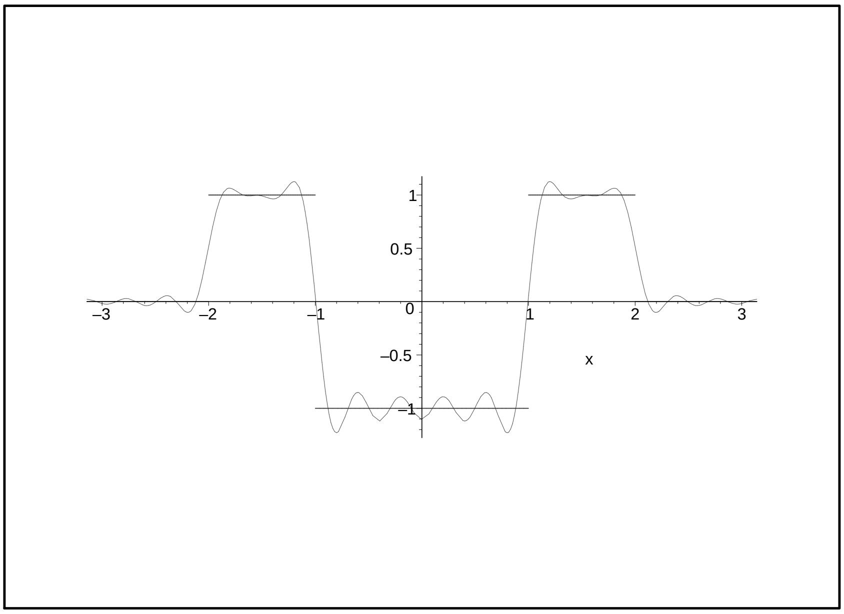


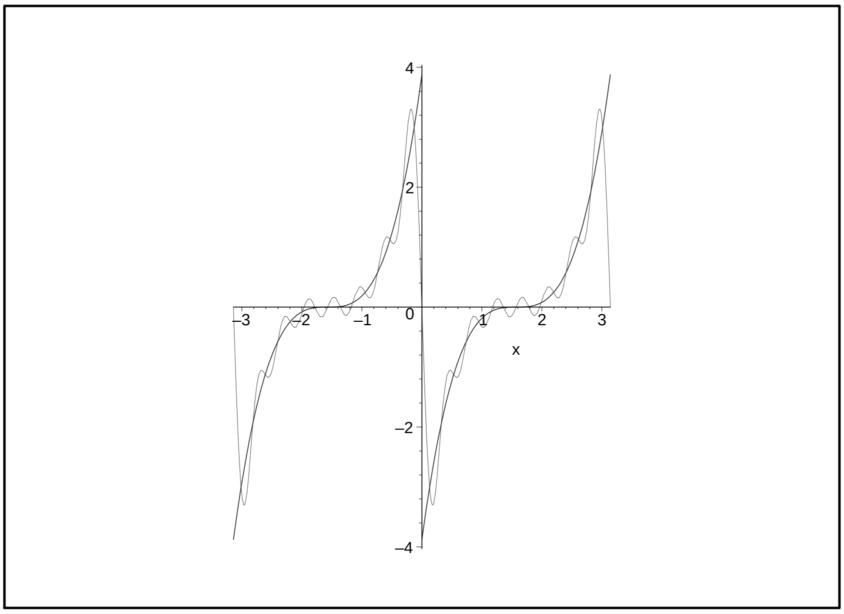
Now consider the function $f(x) = \begin{cases} (y + \pi/2)^3, & x \in [-\pi, 0] \\ (y - \pi/2)^3, & x \in (0, \pi]. \end{cases}$

Here is its graph together with the graph of $(\sigma_{15} f)(x)$.



Just out of curiosity, we plot these two functions together with their Fourier polynomials $(S_{15}f)(x)$ to see if there is any hope.





It looks like there is some hope after all. However the convergence of Fourier polynomials to the required function requires some more restrictions on f.

We are aiming at obtaining a uniform and absolute bound on the coefficients of the Fourier series of a function. In other words we are hoping that some conditions on the function

f will allow us to find numbers A_k and B_k satisfying

$$|a_k(f)\cos kx| \leq A_k,\ \ |b_k(f)\sin kx| \leq B_k$$

such that the series $\sum_{k=1}^{\infty} A_k$ and $\sum_{k=1}^{\infty} B_k$ both converge. This will allow us to use Weierstrass M-test on the Fourier series of f and we will conclude that the Fourier series of f converges uniformly and absolutely. Whether it converges to f or not is another matter which is resolved by Fourier himself.

Fourier's Theorem: If a trigonometric series $\dfrac{a_0}{2} + \displaystyle\sum_{k=0}^{\infty}[a_k \cos kx + b_k \sin kx]$ converges uniformly to a function on $[a, b]$, then it is the Fourier series of that function.

It remains to understand under which conditions we can bound the Fourier coefficients. Therefore it is natural to first consider the growth behavior of the Fourier coefficients. The following two results here show us how different the situation of Fourier series is from that of power series.

Bessel's Inequality: The series $\sum_{k=1}^{\infty}[a_k(f)]^2$ and $\sum_{k=1}^{\infty}[b_k(f)]^2$ converge. (And their sum cannot exceed $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}[f(t)]^2\,dt$.)

As an immediate corollary of this we have:
Riemann-Lebesgue Lemma: The general terms $a_k(f)$ and $b_k(f)$ tend to zero as $k \to \infty$.

These last two results hold for any integrable and periodic function f. This shows that it is not easy to show even the divergence of a Fourier series since the general term already goes to zero no matter what. But the general terms do not go to zero too slowly; their squares always add up to a finite number.

The first significant result about the growth of Fourier coefficients is the following theorem.

Theorem: If $f : \mathbb{R} \to \mathbb{R}$ is $C^N([-\pi, \pi])$ with each $f^{(\ell)}$ periodic, where $\ell = 0, \ldots, N$, then

$$\lim_{k\to\infty} k^N\, a_k = 0, \text{ and } \lim_{k\to\infty} k^N\, b_k = 0.$$

In particular, for all large k, we have

$$|a_k| \leq \frac{1}{k^N}, \text{ and } |b_k| \leq \frac{1}{k^N}.$$

As an immediate corollary we have the following striking result:

Corollary: If f is twice differentiable and periodic, then the Fourier series of f converges absolutely and uniformly to f.

This situation is remarkably different than that of power series since a function can be infinitely differentiable and yet its Taylor series may refuse to converge to the function.

In fact more is true. If a periodic function is continuously differentiable only once, then its Fourier series still converges to the function. This follows from the following strong result.

Dirichlet-Jordan Theorem: If $f : \mathbb{R} \to \mathbb{R}$ is periodic and of bounded variation, then the Fourier series of f converges to $\dfrac{f(x+) + f(x-)}{2}$ at every x. Moreover the convergence is uniform on every closed interval where the function f is continuous.

The final question that still remains in the air is the uniqueness in the case of pointwise convergence. Recall that Fourier's theorem answers this affirmatively when the convergence is uniform. This final issue is settled by the following theorem:

Cantor's Theorem: Suppose a trigonometric series converges pointwise to a periodic and integrable function, then it is the Fourier series of that function.

Miscellaneous:
If f is an even function, then no sine appears in its Fourier series and all $b_k = 0$. Similarly if f is odd, then all $a_k = 0$.

If f is already a polynomial in sine and cosine, then its Fourier series will also be a polynomial, though probably a different one. This is one way to discover numerous trigonometric identities. For example

$$\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3\,x.$$

Even when we do not know if f is of bounded variation, which is very unlikely in real life!, we can say something about the square convergence of the Fourier coefficients. This is known as *Parseval's identity* and states that if f is periodic and continuous then Bessel's inequality becomes an equality:

$$\frac{[a_0(f)]^2}{2} + \sum_{k=1}^{\infty} [a_k(f)]^2 + \sum_{k=1}^{\infty} [b_k(f)]^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(t)]^2 \, dt.$$

What should you be able to do with all this:
You should be able to calculate the Fourier coefficients of a given function. This involves finding a general formula for $a_k(f)$ and $b_k(f)$. You should know if this series converges to f or not. Using this or Parseval's identity you should be able to obtain numerical identities about the series formed by the coefficients or with their squares.

An Example:
Let $f(x) = e^x$ for $x \in [-\pi, \pi]$, and extend f periodically to all of $\mathbb{R}$.

f is piecewise differentiable on $\mathbb{R}$ and its Fourier series converges to f everywhere on $(-\pi, \pi)$, and converges to $\dfrac{e^\pi + e^{-\pi}}{2} = \cosh\pi \approx 11.59$ at $x = \pm\pi$.

We calculate the Fourier coefficients of f as

$$a_k(f) = (-1)^k \frac{2}{1+k^2} \frac{\sinh\pi}{\pi}, \; k = 0, 1, \ldots,$$

and

$$b_k(f) = (-1)^{k+1} \frac{2k}{1+k^2} \frac{\sinh\pi}{\pi}, \; k = 1, 2, \ldots.$$

Hence we have, using the Fourier series of $f(x) = e^x$

$$\frac{\pi}{\sinh\pi} e^x = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2}{1+k^2} \left(\cos kx - k \sin kx\right), \quad \text{for} \quad x \in (-\pi, \pi).$$

Putting $x = 0$, we get

$$\frac{\pi}{\sinh\pi} = \sum_{k=2}^{\infty} (-1)^k \frac{2}{1+k^2} \approx 0.27.$$

Using Parseval's identity, we get

$$\frac{\sinh 2\pi}{\pi} \frac{\pi^2}{\sinh^2\pi} = \frac{2\pi}{\coth 2\pi - \text{cosech} 2\pi} = 2 + \sum_{k=1}^{\infty} \frac{4}{1+k^2} \approx 6.30.$$

The graph of $y = e^x$ together with the graph of $y = (S_5 f)(x)$ on$[-\pi, \pi]$ is below.