Date: January 11, 2010, Monday NAME:....................................................
Time: 15:30-17:30
Ali Sinan Sertöz STUDENT NO:....................................................

**Math 213 Advanced Calculus – Final Exam – Solutions**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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**Q-1)** Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} x^n.$$

**i)** Find the radius of convergence.
**ii)** Decide if the series converges at the end points.
*[10+10 points]*

**Solution:**

$$\begin{aligned} a_n &= \frac{(n!)^3}{(3n)!} x^n. \\ a_{n+1} &= \frac{(n!)^3 (n+1)^3}{(3n)!\,(3n+1)(3n+2)(3n+3)} x^{n+1}. \\ \left|\frac{a_{n+1}}{a_n}\right| &= \left(\frac{n+1}{3n+1}\right)\left(\frac{n+1}{3n+2}\right)\left(\frac{n+1}{3n+3}\right)|x| \\ &= \left(\frac{1+1/n}{3+1/n}\right)\left(\frac{1+1/n}{3+2/n}\right)\left(\frac{1+1/n}{3+3/n}\right)|x|. \end{aligned}$$

$$\text{Then } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3^3}.$$

Therefore the series converges absolutely for $|x| < 27$.

Since $\left(\frac{n+1}{3n+1}\right) > \frac{1}{3}$, $\left(\frac{n+1}{3n+2}\right) > \frac{1}{3}$ and $\left(\frac{n+1}{3n+3}\right) = \frac{1}{3}$, at the end points when $|x| = 27$ we will have

$$\left|\frac{a_{n+1}}{a_n}\right| > 1, \text{ and hence } |a_{n+1}| > |a_n| > \cdots > |a_1| > 0.$$

Then we see that the general term does not converge to zero as $n$ goes to $\infty$. Hence the series diverges at the endpoints.

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**Q-2)** Let $f_n(x) = \left(1 + \frac{x}{n}\right)^n$, where $x \in [0,1]$ and $n = 1, 2, \ldots$

**i)** Show that $f_n(x)$ is increasing as $n$ increases.
You may find Bernoulli's inequality useful: $(1+\delta)^\alpha \leq 1 + \alpha\delta$, for $0 < \alpha \leq 1$ and $\delta > -1$.

**ii)** For each $n$, find the maximum of $h_n(x) = e^x - \left(1 + \frac{x}{n}\right)^n$, where $x \in [0,1]$.
**iii)** Prove or disprove that the sequence $\{f_n(x)\}$ converges to $e^x$ uniformly on $[0,1]$ as $n \to \infty$.
*[7+6+7 points]*

**Solution:**

Using Bernoulli's inequality, we can write

$$\left(1 + \frac{x}{n}\right)^{\frac{n}{n+1}} \leq 1 + \frac{n}{n+1}\frac{x}{n} = 1 + \frac{x}{n+1}.$$

Taking the $n+1$-st power of all sides, we get

$$\left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{x}{n+1}\right)^{n+1},$$

which shows that $f_n(x)$ increases as $n$ increases.

To find the maximum of $h_n(x)$ we take its derivative.

$$h_n'(x) = e^x - \left(1 + \frac{x}{n}\right)^{n-1} > e^x - \left(1 + \frac{x}{n}\right)^n > 0.$$

Thus $h_n(x)$ is strictly increasing and takes its maximum at the right end point $x = 1$.

Using the previous results, we observe that

$$\left|e^x - \left(1 + \frac{x}{n}\right)^n\right| < e - \left(1 + \frac{1}{n}\right)^n = e - f_n(1).$$

We know that $f_n(1)$ converges to $e$ as $n$ goes to infinity. Therefore, for any $\epsilon > 0$, we can find an index $N$ such that for all $n \geq N$, we have $|e - f_n(1)| < \epsilon$. Combining this with the above inequalities, we conclude that the same $N$ works for all $x \in [0,1]$, and hence the convergence is uniform.

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**Q-3)** Let $f(x) = 1 - x^2$ for $x \in [-\pi, \pi]$, and extend $f$ to $\mathbb{R}$ periodically.

**i)** Find all the Fourier coefficients $a_k(f)$ and $b_k(f)$ of $f$.

**ii)** Does the Fourier series of $f$ converge? To what function does it converge? Why?

**iii)** Find the value of the sum $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots$.

*[7+6+7 points]*

**Solution:**

By direct computation we find that $a_0(f) = 2 - \frac{2}{3}\pi^2$, and $a_n(f) = (-1)^{n+1}\frac{4}{n^2}$. Since $f$ is even, all $b_n(f) = 0$.

The Fourier series of $f$ converges uniformly by the Weierstrass M-test, so it converges to $f$, by Fourier's theorem.

Evaluating the identity $f(x) = (Sf)(x)$ at $x = 0$ we get

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \approx 0.8224670336.$$

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**Q-4)** Let $f(x) = |x|$ for $x \in [-\pi, \pi]$, and extend $f$ to $\mathbb{R}$ periodically.

**i)** Find all the Fourier coefficients $a_k(f)$ and $b_k(f)$ of $f$.

**ii)** Does the Fourier series of $f$ converge? To what function does it converge? Why?

**iii)** Find the value of the sum $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots + \frac{1}{(2n-1)^4} + \cdots$.

*[7+6+7 points]*

**Solution:**

Clearly $f$ is of bounded variation, so its Fourier series converges to $f$. This is calculated in the book as an example and we have

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

Since $f$ is continuous, Parseval's identity

$$\frac{a_0(f)^2}{2} + \sum_{n=1}^{\infty} [a_n(f)^2 + b_n(f)^2] = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx$$

gives, after simplifications,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} \approx 1.014678032.$$

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**Q-5)** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that for any $\epsilon > 0$ there exists a (classical) polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$, for all $x \in [a, b]$.
*[20 points]*

**Solution:**
By Fejer's theorem, every continuous function can be uniformly approximated by a Cesaro sum which is a trigonometric polynomial. Being a trigonometric polynomial, a Cesaro sum is an analytic function of $x$ and can be uniformly approximated by Taylor polynomials. Putting these together, we can uniformly approximate $f$.