MATH 213 HOMEWORK
December 2009

Q-1) Let $f_0, \dots, f_n, \dots$ be a sequence of functions such that the series $\sum_{n=0}^{\infty} f_n$ converges to a function f. Let

$$(f_0 + \cdots + f_{n_1}) + (f_{n_1+1} + \cdots + f_{n_2}) + \cdots$$

be any pattern of inserting parenthesis into this infinite sum without changing the places of any f_n. Show that the new series with the parenthesis also converges to f. Use this to show that

$$a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots = \sum_{k=0}^{\infty} \left(a_k \cos kx + b_k \sin kx \right),$$

whenever the series converge.

Solution: Let $s_N = \sum_{n=0}^{N} f_n$ be the sequence of partial sums converging to f. Putting parenthesis amounts to choosing a subsequence of the sequence s_n, so the new subsequence also converges to f.

Applying to the trigonometric series above is now straightforward.

Q-2) Let $f(x)$ be a function on $\mathbb{R}$, which is C^∞ and periodic with period 2π. Show that all derivatives of f are also periodic with period 2π. Using integration by parts twice in the calculation of the Fourier coefficients of f, show that the Fourier series of f converges uniformly and absolutely to f on $\mathbb{R}$.

Solution: Since the graph of f repeats on blocks of $[-\pi, \pi]$, it is clear that all properties of f will repeat and hence be periodic.

Let M be an upper bound for $f''(x)$ for $x \in [-\pi, \pi]$. Here is how we calculate a_k:

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \\
&= \left(\frac{f(x)\sin kx}{k\pi}\Bigg|_{-\pi}^{\pi}\right) - \frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin\ kx dx, \quad \text{(using by-parts)} \\
&= -\frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin kx\, dx \\
&= \left(\frac{f'(x)\cos kx}{k^2\pi}\Bigg|_{-\pi}^{\pi}\right) - \frac{1}{k^2\pi}\int_{-\pi}^{\pi} f''(x)\cos kx\, dx, \quad \text{(using by-parts)} \\
&= -\frac{1}{k^2\pi}\int_{-\pi}^{\pi} f''(x)\cos kx dx, \quad \text{since } f' \text{ is periodic.} \\
|a_k| &= \frac{1}{k^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos kx\, dx\right| \\
&\le \frac{2M}{k^2}, \quad k = 1, 2, \ldots
\end{aligned}$$

Similarly for $|b_k|$. Using Weierstrass M-test we conclude that the Fourier series converges uniformly and absolutely, in which case it should converge to $f(x)$.

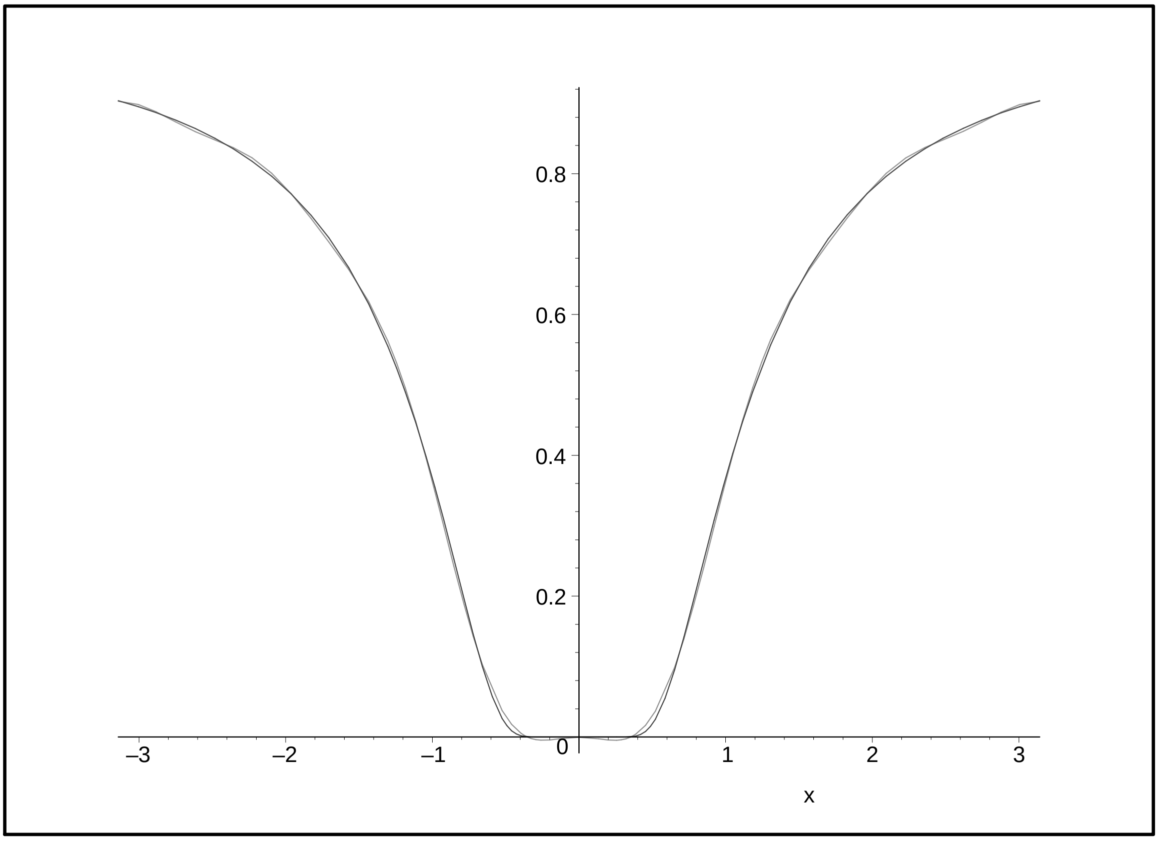
Q-3) Consider the function f which is defined on $[-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} exp(-1/x^2) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Use a computer algebra system to numerically write the Fourier sum $S_6 f$ of f. Plot both f and $S_6 f$ on the same graph.

Solution:

$S_6 f = 0.5354546575 - 0.4546891906\, \cos(x) - 0.1214137143\, \cos(2\,x) - 0.01696975846\, \cos(3\,x) + 0.02218794214\, \cos(4\,x) + 0.02005516168\, \cos(5\,x) + 0.01498074181\, \cos(6\,x)$



As you can see, the graphs of f and $S_6 f$ are almost indistinguishable. Recall that f is not analytic and in fact its Taylor series around $x = 0$ is identically zero and hence has nothing to do with f.

Q-4) Find the Fourier series of $f(x) = |\cos x|$. Where does it converge to $f(x)$? Why? Using this Fourier series give an infinite numerical series which converges to π.

Solution:

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos 2kx, \quad \text{for } x \text{ in any closed interval in } (-\pi,\pi).$$

It is easy to see that $|\cos x|$ is of bounded variation, so its Fourier series converges to the function everywhere.

Putting $x = 0$ in the Fourier series above we get

$$\pi = 2 + 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^2-1}.$$

Q-5) Let $f(x) = |x|$ on $[-\pi, \pi]$ and extend it to $\mathbb{R}$ periodically. Using the Fourier series of f find two numerical series, one converging to π^2 and one converging to π^4.

Solution:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x.$$

Putting $x = 0$ we find

$$\pi^2 = 8 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Using Parseval equality (since f is continuous on $\mathbb{R}$) we get

$$\pi^4 = 96 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}.$$

Comments and questions to `sertoz@bilkent.edu.tr`