Date: October 15, 2009, Thursday NAME:..
Time: 13:40-15:30
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Math 213 Advanced Calculus I – Midterm Exam I – Solutions

Q-1) Let $r_n = \dfrac{2 \cdot 4 \cdot 6 \cdots\cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \alpha^n$, where $n = 1, 2, 3, \ldots$ and $\alpha > 0$ is a real constant.

Find $\lim_{n\to\infty} r_n$ for all possible values of α.

Solution: For the case $0 < \alpha < 1$, first observe that $\dfrac{r_{n+1}}{r_n} = \dfrac{2n+2}{2n+1} \cdot \alpha \to \alpha$ as $n \to \infty$, so the series $\sum_{n=1}^{\infty} r_n$ converges, forcing the general term r_n to go to zero as n goes to infinity.

For the case $\alpha \geq 1$, observe that $r_n = (2^n)\left(\dfrac{1}{1}\,\dfrac{2}{3}\,\dfrac{3}{5}\cdots\dfrac{n}{2n-1}\right)\alpha^n > (2^n)\left(\dfrac{2^{n-1}}{3^{n-1}}\right)\alpha^n = \left(\dfrac{3}{2}\right)\left(\dfrac{4}{3}\right)^n \alpha^n \to \infty$ as $n \to \infty$.

Conclusion: $\lim_{n\to\infty} r_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1, \\ \infty & \text{if } \alpha \geq 1. \end{cases}$

Another way to see $\lim_{n\to\infty} r_n = 0$ for $0 < \alpha < 1$ is as follows:

Choose β with $\alpha < \beta < 1$. Since $\lim_{k\to\infty} \frac{2k}{2k-1} = 1$ and since $\frac{2k}{2k-1} > 1$, we can find N such that for all $k > N$, $\frac{2k}{2k-1} < \frac{1}{\beta}$.

Now for all $n > N$, $r_n = \left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)\alpha^n = \left(\prod_{k=1}^{N} \frac{2k}{2k-1}\right)\left(\prod_{k=N+1}^{n} \frac{2k}{2k-1}\right)\alpha^n < \left(\prod_{k=1}^{N} \frac{2k}{2k-1}\right)(\beta^N)\left(\frac{\alpha}{\beta}\right)^n$ which goes to zero as $n \to \infty$, since $0 < \frac{\alpha}{\beta} < 1$.

Another neat solution by Ömer Faruk Tekin: $\lim_{n\to\infty} \dfrac{r_{n+1}}{r_n} = \alpha$. So $\forall \epsilon > 0\ \exists N$ such that $\forall n \geq N$,

$$(\alpha - \epsilon)\, r_n < r_{n+1} < (\alpha + \epsilon)\, r_n.$$

When $\alpha > 1$, choose $\epsilon > 0$ such that $1 < (\alpha - \epsilon)$. Then

$$r_{N+m} > (\alpha - \epsilon)^m\, r_N, \quad \text{so} \quad \lim_{n\to\infty} r_n = \infty.$$

When $\alpha = 1$, show by induction that $r_n > \sqrt{n+1}$. Then again $\lim_{n\to\infty} r_n = \infty$.

When $0 < \alpha < 1$, choose $\epsilon > 0$ such that $(\alpha + \epsilon) < 1$. Then

$$r_{N+m} < (\alpha + \epsilon)^m\, r_N, \quad \text{so} \quad \lim_{n\to\infty} r_n = 0.$$

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Q-2) Find the sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}$.

Solution: $\displaystyle\frac{1}{(n+1)(n+2)(n+3)} = \frac{1/2}{n+1} - \frac{1}{n+2} + \frac{1/2}{n+3}$.

Let $\displaystyle s_k = \sum_{n=0}^{k} \frac{1}{(n+1)(n+2)(n+3)}$. Then we have

$$\begin{aligned}
s_k &= \frac{1/2}{1} - \frac{1}{2} + \frac{1/2}{3} \\
&\quad + \frac{1/2}{2} - \frac{1}{3} + \frac{1/2}{4} \\
&\quad + \frac{1/2}{3} - \frac{1}{4} + \frac{1/2}{5} \\
&\quad \vdots \\
&\quad + \frac{1/2}{k+1} - \frac{1}{k+2} + \frac{1/2}{k+3} \\
&= \frac{1}{4} - \frac{1/2}{k+2} + \frac{1/2}{k+3}, \quad \text{after cancellations.}
\end{aligned}$$

So the sum is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} = \lim_{k\to\infty} s_k = \frac{1}{4}$.

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Q-3) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}\,(x-\pi)^n$.

(Do not forget to check both end points.)

Hint: $(1+1/n)^n < e$ for all $n \geq 1$.

Solution: Use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^n\,|x-\pi|}{(n+1)^n} = \frac{|x-\pi|}{(1+1/n)^n} \to \frac{|x-\pi|}{e} \text{ as } n \to \infty.$$

The radius of convergence is e.

At the end points $|x-\pi| = e$. In that case $\left|\frac{a_{n+1}}{a_n}\right| = \frac{e}{(1+1/n)^n} > 1$, so the general term does not go to zero, hence the series diverges at the end points of the interval.

Conclusion: The interval of convergence is $(\pi - e, \pi + e)$.

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Q-4) Find $\lim_{x\to 0} \dfrac{x\,\ln(1+x)\,\sin(x^2)}{e^{x^2}+\cos(\sqrt{2}\,x)-2}$.

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{x\,\ln(1+x)\,\sin(x^2)}{e^{x^2}+\cos(\sqrt{2}\,x)-2} &= \lim_{x\to 0} \frac{x\,(x-x^2/2+x^3/3+\cdots)\,(x^2-x^6/6+\cdots)}{(1+x^2+x^4/2+\cdots)+(1-x^4+x^4/6-\cdots)-2} \\ &= \lim_{x\to 0} \frac{x^4+x^5(*)}{\frac{2}{3}x^4+x^5(*)} \\ &= \lim_{x\to 0} \frac{1+x(*)}{\frac{2}{3}+x(*)} \\ &= \frac{3}{2}.\end{aligned}$$

If you want to do this with L'Hopital's rule, then you have to take fourth derivatives. The fourth derivative of the numerator is:

$$2\Bigg(12\cos(x^2)-132\,x^3\sin(x^2)-16\,x^4\cos(x^2)+8\,x^2\cos(x^2)+4\sin(x^2)+18\cos(x^2)\,x-$$
$$180\,x^3\ln(x+1)\sin(x^2)-120\ln(x+1)\sin(x^2)\,x^2-160\,x^4\ln(x+1)\cos(x^2)-40\ln(x+1)\cos(x^2)\,x^3-$$
$$30\,x\ln(x+1)\sin(x^2)+2\cos(x^2)\,x^3-36\sin(x^2)\,x^5-120\sin(x^2)\,x^4-16\,x^7\cos(x^2)-$$
$$48\,x^6\cos(x^2)-48\,x^5\cos(x^2)+\sin(x^2)\,x-22\,x^5\ln(x+1)\sin(x^2)-160\ln(x+1)\cos(x^2)\,x^6-$$
$$240\ln(x+1)\cos(x^2)\,x^5-120\,x^4\ln(x+1)\sin(x^2)+8\,x^9\ln(x+1)\sin(x^2)+32\,x^8\ln(x+1)\sin(x^2)+$$
$$48\,x^7\ln(x+1)\sin(x^2)+32\,x^6\ln(x+1)\sin(x^2)-48\sin(x^2)\,x^2-40\ln(x+1)\cos(x^2)\,x^7\Bigg)/(x+1)^4$$

and its value at $x=0$ is 24.

The fourth derivative of the denominator is:

$$12\,e^{x^2}+48\,x^2e^{x^2}+16\,x^4e^{x^2}+4\cos\left(\sqrt{2}x\right)$$

and its value at $x=0$ is 16. Thus giving the limit value as $24/16=3/2$.

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Q-5) Find the sum $\sum_{n=0}^{\infty}(-1)^n \frac{n}{3^n}$.

Solution:

Let $f(x) = x \cdot (\frac{d}{dx}\frac{1}{1+x}) = \frac{-x}{(1+x)^2} = -x + 2x^2 - 3x^3 + \cdots + (-1)^n n x^n + \cdots$.

Then $f(1/3) = \sum_{n=0}^{\infty}(-1)^n \frac{n}{3^n}$ and clearly $f(1/3) = -\frac{3}{16}$.