Date: November 19, 2009, Thursday
Time: 13:40-15:30
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NAME:..

STUDENT NO:..

Math 213 Advanced Calculus I – Midterm Exam II – Solutions

Q-1) Define $T_n(x) = x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, where n is a positive integer and x is a real number. Show that

$$1 + T_5(x) \leq e^x \leq \frac{241}{240} + T_5(x), \quad \text{for all } x \in [0,1].$$

When do we have, if ever, equalities on either side?

Solution: Taylor's theorem states that

$$e^x = 1 + T_5(x) + \frac{e^c}{6!}x^6, \text{ for some } c \text{ between } x \text{ and } 0.$$

In our case c must be somewhere in $[0,1]$. Taking $e < 3$, we find that for $x \in [0,1]$,

$$0 \leq \frac{e^c}{6!}x^6 < \frac{3}{6!} = \frac{1}{240}.$$

Adding $1 + T_5(x)$ to all sides of this inequality, we obtain the claimed result.

The first inequality becomes an equality when $x = 0$. Since e^x is strictly increasing, for $x > 0$, the first inequality is always strict. The second inequality is always strict on $[0,1]$ because we replaced e by 3 in the error estimate.

NAME: STUDENT NO:

Q-2) Show that $\cos 1$ is irrational.

Solution: We know that

$$\cos 1 = 1 - \frac{1}{2!} + \cdots + \frac{(-1)^k}{(2k)!} + \cdots .$$

Assume to the contrary that $\cos 1$ is rational. Let $\cos 1 = \frac{p}{q}$ for some integers p and q. ($\cos 1 =$ .5403023059..., but we will not use this information.) Let n be an odd integer such that $2n > |q|$. We are assuming that

$$\frac{p}{q} = 1 - \frac{1}{2!} + \cdots + \frac{(-1)^k}{(2k)!} + \cdots .$$

Multiply both sides by $(2n)!$ to obtain

$$\frac{p\,(2n)!}{q} = (2n)!\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^n}{(2n)!}\right) + (2n)!\left(\frac{(-1)^{n+1}}{(2n+2)!} + \frac{(-1)^{n+2}}{(2n+4)!} + \cdots\right),$$

or equivalently (recalling that n is odd)

$$\frac{p\,(2n)!}{q} - (2n)!\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^n}{(2n)!}\right) = (2n)!\left(\frac{1}{(2n+2)!} - \frac{1}{(2n+4)!} + \cdots\right).$$

Observe that the left hand side is an integer. Call it N. Simplifying the right hand side we obtain

$$N = \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)(2n+4)} + \cdots$$

Since the right hand side is an alternating sum and the general term decreases steadily down to zero, we must have

$$0 < N < \frac{1}{(2n+1)(2n+2)} < 1.$$

But there is no integer in the interval $(0, 1)$. This contradiction shows that $\cos 1$ is irrational.

NAME: STUDENT NO:

Q-3) Define $f_n(x) = x/n$ and $f(x) = 0$, where n is a positive integer and x is a real number. Show that $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise on any interval $I \subset \mathbb{R}$.
Is this convergence uniform when
(i) $I = [0, 1]$?
(ii) $I = \mathbb{R}$?

Solution: For any $x \in \mathbb{R}$, x/n goes to 0 as n goes to ∞. So we have the pointwise convergence on every interval.

Let $I = [0, 1]$. For any $\epsilon > 0$, choose an integer $N > 1/\epsilon$. Then for any $n \geq N$ and for any $x \in [0, 1]$, we have

$$|f_n(x) - f(x)| = \frac{x}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

So the convergence is uniform on $[0, 1]$.

Let $I = \mathbb{R}$. Take $\epsilon = 1$. For every integer $N > 0$ take $n = N$ and $x_0 \geq N\epsilon = N$. Then we have

$$|f_n(x_0) - f(x_0)| = \frac{x_0}{n} = \frac{x_0}{N} \geq \epsilon = 1.$$

So the convergence is not uniform on $\mathbb{R}$.

NAME: STUDENT NO:

Q-4) Construct a sequence $\{f_n(x)\}$ of continuous functions on $[0, 1]$ which converges pointwise but not uniformly to a continuous function $f(x)$ on $[0, 1]$.

Solution: Let $f(x) = 0$ for all x. The idea then, is to let the graph of $f_n(x)$ to be zero everywhere except on the interval $[1/(2n), 1/n]$ where it is a triangle of fixed height. The following description achieves this.

For any two real numbers $a < b$, define a function $\phi_{[a,b]}(x)$ as follows

$$\phi_{[a,b]}(x) = \begin{cases} \frac{2}{b-a} \min\{x-a, b-x\} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

The graph of $\phi_{[a,b]}(x)$ has a triangle of fixed height 1 on the interval $[a, b]$ and is zero elsewhere.

Now define the sequence as $f_n(x) = \phi_{[1/(2n),1/n]}(x)$ for $x \in [0, 1]$.

For any $x \in (0, 1]$, let N_x be an integer with $N_x > 1/x$, and let $N_0 = 1$. Then for any $x \in [0, 1]$ and for any $n \geq N_x$, we have $f_n(x) = 0$, which gives the poinwise convergence.

But the convergence is not uniform; Take $0 < \epsilon \leq 1$. For any N, take $n \geq N$ and take $x = 3/(4n)$, the midpoint of the interval $[1/(2n), 1/n]$. Then $|f_n(x) - f(x)| = 1 \geq \epsilon$. So the convergence is not uniform.

NAME: STUDENT NO:

Q-5) Is it possible to construct a sequence $\{f_n(x)\}$ of differentiable functions on $[-1, 1]$, which converges uniformly on that interval to the absolute value function $f(x) = |x|$ (which we know is not differentiable at $x = 0$)? If *yes*, construct such a sequence. If *not*, explain why.

Solution: The example given in class by the back-seat-gang works perfectly well for this problem. Here is their solution.

Let $f_n(x) = \sqrt{x^2 + 1/n}$. This clearly converges to the absolute value function as n goes to ∞.

To show that the convergence is uniform, for any $\epsilon > 0$ take $N > 1/\epsilon^2$. Now for any $n \geq N$ and any $x \in [-1, 1]$, we have

$$|f_n(x) - f(x)| = \sqrt{x^2 + 1/n} - \sqrt{x^2} = \frac{1/n}{\sqrt{x^2 + 1/n} + \sqrt{x^2}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon,$$

which establishes the uniform convergence.

Here is the graph of $|x|$ together with $f_n(x)$ for $n = 10, 20, 200$.