Math 214 Advanced Calculus II
Final Exam
SOLUTIONS

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1) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that all directional derivatives of f at the origin exist but f is not differentiable at the origin. Is it possible to construct this f such that it is differentiable at the origin but some directional derivatives at the origin do not exist? Explain.

Solution: The first part is Exercise 3-b on page 295 and solved in detail in class. The second part is the content of Theorem 6.7 on page 288.

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2) Consider the functions

$$\begin{aligned} F(t) &= (5t^3 + 7t^2 + 12t + 13, \lambda t - 9) \\ G(x,y) &= x^2 + y^2 + 3xy + 2x - 18y + 1. \end{aligned}$$

Find those λ for which
(i) $F \circ G$ is invertible around the origin.
(ii) $G \circ F$ is invertible around the origin.

Solution:

$$DF(t) = \begin{pmatrix} 15t^2 + 14t + 12 \\ \lambda \end{pmatrix}, \quad DG(x,y) = (2x + 3y + 2, 2y + 3x - 18)\,.$$

(i) $G(0,0) = 1$, so $DG(0,0) = (2, -18)$ and $DF(1) = \begin{pmatrix} 41 \\ \lambda \end{pmatrix}$. Hence $D(F \circ G)(0,0) = DF(1) \cdot DG(0,0) = \begin{pmatrix} 82 & -738 \\ 2\lambda & -18\lambda \end{pmatrix}$. Since $detD(F \circ G)(0,0) = 0$, the function $F \circ G$ is not invertible around the origin for any value of λ.
(ii) $F(0) = (13, -9)$ and $DF(0) = \begin{pmatrix} 12 \\ \lambda \end{pmatrix}$, $DG(13, -9) = (1, 3)$. Hence $D(G \circ F)(0) = DG(13, -9) \cdot DF(0) = 12 + 3\lambda$. Thus the function $G \circ F$ is invertible around the origin for all values of $\lambda \neq -4$.

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3) Let $\mathbf{v}_j = (v_{j1}, \ldots, v_{jn}) \in \mathbb{R}^n$, $j = 1, \ldots, n$, be fixed. The parallelpiped determined by the vectors $\mathbf{v}_j$'s is the set

$$\mathcal{P} := \{t_1\mathbf{v}_1 + \cdot + t_n\mathbf{v}_n \mid t_j \in [0,1]\ \}.$$

Prove that

$$Vol(\mathcal{P}) = |\det(v_{ij})_{n\times n}|.$$

Solution: Consider the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi(x_1, \ldots, x_n) = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$. Let $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_1, \ldots, x_n \le 1\ \}$ be the unit n-cube in $\mathbb{R}^n$. Then $\phi(C) = \mathcal{P}$ and

$$Vol(\mathcal{P}) = \int_{\phi(C)} du = \int_C |\Delta_\phi(x)| dx.$$

Since $\Delta_\phi(x) = \det(v_{ij})_{n\times n}$, the result follows.

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4) Let

$$F(x,y,z) = \left(\frac{x}{(3x^2+5y^2+7z^2)^{3/2}}, \frac{y}{(3x^2+5y^2+7z^2)^{3/2}}, \frac{z}{(3x^2+5y^2+7z^2)^{3/2}} \right).$$

Show that $divF = 0$. Show that there is no vector field $G(x,y,z)$ such that $curlG = F$.

Solution: This is almost the same as Remark 2 on page 451. That $divF = 0$ is straightforward. The rest of the solution follows exactly the same lines of proof on page 452 except that you have to observe that for this problem $\displaystyle\int_S F \cdot \mathbf{n} d\sigma$ is not easy to evaluate. However you can easily observe that it must be positive. This then supplies the required contradiction.

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5) Let $\omega = (y+z)dxdy + (x+z)dydz + (x+y)dzdx$ be a 2-form on $\mathbb{R}^3$, and let S be the unit sphere centered at the origin in $\mathbb{R}^3$. Evaluate

$$\int_S \omega.$$

In the above integral there is a difference between $dxdy$ and $dydx$. However Fubini's theorem tells us that we can integrate with respect to any order. How do you explain this *inconsistency*?

Solution: Let B be the unit ball centered at the origin in $\mathbb{R}^3$.
$d\omega = (dy+dz)dxdy + (dx+dz)dydz + (dx+dy)dzdx = 3dxdydz.$

$$\int_S \omega = \int_B d\omega = \int_B 3dxdydz = 3Vol(B) = 4\pi.$$

In the Stokes theorem there is a difference between $dxdy$ and $dydx$. This tells us which integral is equal to which integral. But Fubini's theorem is about evaluating the integral. Once you decide which integral to evaluate, say using Stokes theorem, then Fubini's theorem says that you can integrate it as an iterated integral in any order you like, under certain mild conditions. Stokes Theorem: page 515. Fubini's Theorem: page 358.
