Math 214 Advanced Calculus II
Midterm Exam I

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1) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f_x and f_y exist everywhere on $\mathbb{R}^2$ but the function f itself is not differentiable at $(0,0)$. Prove however that if you further assume that f_x and f_y are continuous at the origin, then f has to be differentiable there.

Solution: Such an example is given in Example 2 on page 291. The proof that continuity of the partials implying differentiability is the content of Theorem 6.9 on page 292.

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2) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a $\boldsymbol{C^2}$ function. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be points in $\mathbb{R}^3$ and define $H(\boldsymbol{x}) = (\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}))_{1\leq i,j\leq 3}$ for $\boldsymbol{x} \in \mathbb{R}^3$. We have the following data: $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$, $\nabla f(\boldsymbol{b}) = \boldsymbol{0}$, $\nabla f(\boldsymbol{c}) = \boldsymbol{0}$ and

$$H(\boldsymbol{a}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -13 & 5 \\ 0 & 5 & -2 \end{pmatrix}, \quad H(\boldsymbol{b}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, H(\boldsymbol{c}) = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

With this much information can you decide which of these points $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are min/max or saddle points for f? If so, classify these points accordingly.

Solution: The criteria to make such decisions is given in Theorem 6.22 on page 321. For the matrix $H = (h_{ij})_{1\leq i,j\leq 3}$ define $H_1 = h_{11}$, $H_2 = h_{11}h_{22} - h_{12}h_{21}$ and $H_3 = det(H)$.
For the first matrix we have $H_1 = -1$, $H_2 = 13$, $H_3 = -1$. Then this is a negative definite matrix and satisfies part (ii) of Theorem 6.22. Thus the point $\boldsymbol{a}$ gives local maximum for f.
The second matrix clearly satisfies the condition (iii) of this theorem so the point $\boldsymbol{b}$ gives a saddle point.
For the third matrix $H_1 = 3$, $H_2 = 5$, $H_3 = 2$. This is then a positive definite matrix and satisfies condition (i) of the theorem and thus $\boldsymbol{c}$ gives a local minimum point for f.

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3) Consider the functions $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by
$f(u,v,w) = (u^3 + v^2 + 2u + 2v + w,\ u^7 + v^6 + w^8 + u + 2v - w)$,
$g(x,y) = (x^5 + xy + y^3 + x + 2y,\ x^5y^7 + y^7 + 3x + 4y,\ x^{21} + y^{30} + 3x + 2y)$.
Calculate $D(f \circ g)(0,0)$.

Solution: Note that $g(0,0) = (0,0,0)$. Let $f = (f_1, f_2)$ and $g = (g_1, g_2, g_3)$.

Then

$$Df(0,0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial u}(0,0,0) & \frac{\partial f_1}{\partial v}(0,0,0) & \frac{\partial f_1}{\partial w}(0,0,0) \\ \\ \frac{\partial f_2}{\partial u}(0,0,0) & \frac{\partial f_2}{\partial v}(0,0,0) & \frac{\partial f_2}{\partial w}(0,0,0) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} = A$$

and

$$Dg(0,0) = \begin{pmatrix} \frac{\partial g_1}{\partial x}(0,0) & \frac{\partial g_1}{\partial y}(0,0) \\ \\ \frac{\partial g_2}{\partial x}(0,0) & \frac{\partial g_2}{\partial y}(0,0) \\ \\ \frac{\partial g_3}{\partial x}(0,0) & \frac{\partial g_3}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 3 & 2 \end{pmatrix} = B.$$

Now $D(f \circ g)(0,0) = AB = \begin{pmatrix} 11 & 14 \\ 4 & 8 \end{pmatrix}$.

The theory behind this is in Theorem 6.13 on page 298.

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4) Let $F(y) = \int_0^\infty e^{-x^2y^2}dx$, $y \in [1,100]$. Calculate $F'(1)$ and justify your answer.
(You may want to be reminded that $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$)

Solution: First assume that $F'(y)$ converges uniformly on $[1,100]$. Then we can apply Theorem 6.6 on page 284 and differentiate under the integral sign:

$$\begin{aligned} F'(1) &= \textstyle\int_0^\infty (-2yx^2e^{-x^2y^2})|_{y=1}dx \\ &= -2\textstyle\int_0^\infty x^2e^{-x^2}dx \\ &= xe^{-x^2}|_{x=0}^{x=\infty} - \textstyle\int_0^\infty e^{-x^2}dx \\ &= -\tfrac{\sqrt{\pi}}{2}, \end{aligned}$$

where we used integration by parts with $u = x$ and $dv = xe^{-x^2}dx$ for integration and used the hint in the last line.
Now for the justification observe that when $y \in [1,100]$,
$|-2yx^2e^{-x^2y^2}| = 2|yx^2|\ |e^{-x^2y^2}| \le 2|100x^2|\ |e^{-x^2}| = 200x^2e^{-x^2}$ And this function is improperly integrable on $[0,\infty)$ as we showed above. Thus by Weierstrass M-Test, on page 283, the integral $F'(y)$ converges uniformly. This satisfies the main requirement of Theorem 6.6 on page 284 and differentiation under the integral sign is justified.

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5) Prove that there exist functions $u(x,y)$, $v(x,y)$ and $w(x,y)$ and $r > 0$ such that u, v, w are continuously differentiable and satisfy the equations

$$\begin{aligned} u^5 + xv^2 - y + w &= 0 \\ v^5 + yu^2 - x + w &= 0 \\ w^4 + y^5 - x^4 &= 1 \end{aligned}$$

on $B_r(1,1)$, and $u(1,1) = 1$, $v(1,1) = 1$, $w(1,1) = -1$.

Solution: This is Exercise 5 on page 318, and I solved it in detail in class. Here we basically use the Implicit Function Theorem on page 315.
