Date: May 25, Tuesday
Time: 15:30-17:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 214 Advanced Calculus – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let M be a compact C^∞ manifold in $\mathbb{R}^n$ of dimension m with the property that for every $x \in M$ there is a coordinate chart (U, ϕ_U) such that $x \in U$ and $\phi_U(U)$ is the unit ball in $\mathbb{R}^n$ centered at the origin.
Define what it means for a function $f : M \to \mathbb{R}$ to be C^∞.
Show that there exists a non-constant C^∞ function $f : M \to \mathbb{R}$ such that $|f(x)| < 1$ for all $x \in M$.

Solution:
The function $f : M \to \mathbb{R}$ is called C^∞ if $f \circ \phi_i^{-1}$ is C^∞ for each i.

The above open sets U constitute an open cover of M and since M is compact, a finite subcover exists, say $U_1, \ldots, U_m$. Let ϕ_i denote ϕ_{U_i}, and let S_i denote the unit disk $\phi_i(U_i)$. By Urysohn's lemma we know that there exists a C^∞ function $h_i : \mathbb{R}^m \to \mathbb{R}$ such that $0 \leq h(x) \leq 1$ for all $x \in \mathbb{R}^m$, $h(x) = 1$ for all $x \in B_{1/2}(0) \subset S_i$ and with support in S_i.

For $x \in M$ define $f_\alpha(x) = (\alpha h_1 \circ \phi_1(x) + h_2 \circ \phi_2(x) + \cdots + h_m \circ \phi_m(x))\,/m$, where $0 < \alpha \leq 1$. At most one of the f_α can be a constant function. All the other f_α satisfy the requirements.

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Q-2) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function.
Define, in ϵ-δ notation, continuity of f on $\mathbb{R}^n$.
Define what it means for f to be differentiable on $\mathbb{R}^n$.
Prove or disprove that f is continuous on $\mathbb{R}^n$ if and only if $f^{-1}(U)$ is open in $\mathbb{R}^n$ for every open U in $\mathbb{R}^m$.

Solution:

f is continuous at $x_0 \in \mathbb{R}^n$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $x \in B_\delta(x_0)$ implies $f(x) \in B_\epsilon(f(x_0))$.

f is continuous on $\mathbb{R}^n$ if it is continuous at every x_0 in $\mathbb{R}^n$.

f is differentiable at $x_0 \in \mathbb{R}^n$ if there exists a linear map $T_{x_0} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - T_{x_0}(x - x_0)}{|x - x_0|} = 0.$$

f is differentiable on $\mathbb{R}^n$ if it is differentiable at every x_0 in $\mathbb{R}^n$.

The statement is true:

First assume that f is continuous. Let U be open in $\mathbb{R}^m$ and let $x_0 \in f^{-1}(U)$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset U$. By continuity of f at x_0, there exists a $\delta > 0$ such that for all $x \in B_\delta(x_0)$, we have $f(x) \in B_\epsilon(f(x_0)) \subset U$. In other words $B_\delta(x_0) \subset f^{-1}(U)$ proving that $f^{-1}(U)$ is open.

Now assume that $f^{-1}(U)$ is open in $\mathbb{R}^n$ for every open U in $\mathbb{R}^m$. Let x_0 be any fixed point in $\mathbb{R}^n$. Choose any $\epsilon > 0$. By assumption $f^{-1}(B_\epsilon(f(x_0))$ is open. In particular there exists a $\delta > 0$ such that $B_\delta(x_0)$ lies in $f^{-1}(B_\epsilon(f(x_0))$, which is the ϵ-δ description of the continuity of f at x_0. Since x_0 was arbitrary, f is continuous everywhere on $\mathbb{R}^n$.

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Q-3) State the implicit function theorem and show how it applies to show that there exists continuously differentiable functions $u(x, y)$ and $v(x, y)$ near the point $(x, y) = (1, 2)$ satisfying the conditions

$$\begin{aligned} xu^2 + yv^2 + xy &= 43 \\ xv^2 + yu^2 - xy &= 32 \end{aligned}$$

and $u(1, 2) = 3$, $v(1, 2) = 4$.

Solution:
The implicit function theorem states that if $F = (F_1, \ldots, F_n)$ is a C^1-function from an open set U in $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ such that for some point $(a, b) \in \mathbb{R}^{n+k}$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$ we have $F(a, b) = 0$ and

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}(a, b) \neq 0,$$

then the variables $x_1, \ldots, x_n$ can be solved in a continuously differentiable way in terms of the remaining variables around the point $b \in \mathbb{R}^k$. In other words, there exists a unique C^1-function g defined from some open neighborhood of $b \in \mathbb{R}^k$ to $\mathbb{R}^n$, $g = (g_1, \ldots, g_n)$, such that $g(b) = a$ and $x_i = g_i(x_{n+1}, \ldots, x_{n+k})$, $i = 1, \ldots, n$.

In the given problem we take $n = k = 2$, $a = (3, 4)$, $b = (1, 2)$ and $(x_1, \ldots, x_4) = (u, v, x, y)$ and $F = (F_1, F_2) = (xu^2 + yv^2 + xy - 43,\ xv^2 + yu^2 - xy - 32)$. Then

$$\frac{\partial(F_1, F_2)}{\partial(u, v)}(3, 4, 1, 2) = \left.\begin{vmatrix} 2xu & 2yv \\ 2yu & 2xv \end{vmatrix}\right|_{(3,4,1,2)} = \left(4uv(x^2 - y^2)\right)_{(3,4,1,2)} = -144 \neq 0,$$

hence u and v can be solved as required in terms of x and y by the implicit function theorem.

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Q-4) Suppose V is open in $\mathbb{R}^n$ and $f : V \longrightarrow \mathbb{R}$ is a continuous function. Assume that

$$\int_E f = 0$$

for every non-empty Jordan region $E \subset V$. Prove or disprove that $f = 0$ on V.

Solution:

The statement is true.

If $f = 0$ on V, then there is nothing to prove. Assume there is $a \in V$ such that $f(a) \neq 0$. Since V is open and since f is continuous, there exists an open ball $B \subset V$ containing a such that $f(x) \neq 0$ for all $x \in B$. Let E be a closed ball in B containing a. Note that E, being a ball in $\mathbb{R}^n$, is a Jordan region. Also note that E is compact and that f is nonzero at every point on E.

By the Mean Value Theorem for Multiple Integrals, there exists a number c such that

$$\inf_{x \in E} \leq c \leq \sup_{x \in E}$$

with the property that

$$c\, Vol(E) = \int_E f.$$

By our assumption, the integral is zero, so $c\, Vol(E) = 0$, forcing $c = 0$. But f being continuous on the compact ball E, there exists a point $x_0 \in E$ such that $f(x_0) = c = 0$ contradicting the way we chose E. Thus there is no point $a \in V$ with $f(a) \neq 0$.

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Q-5) Let ω be a differential r-form.

(i) Prove or disprove: $\omega^2 = 0$ if r is odd.

(ii) Prove or disprove: $\omega^2 = 0$ if r is even.

Solution: Let $\omega = \sum \omega_i$ where $\omega_i = f_i dx_{i_1} \cdots dx_{i_r}$ and f_i is a function. Then $\omega^2 = \sum \omega_i \omega_j = \sum_{i<j}(\omega_i\omega_j + \omega_j\omega_i)$.

Observe that $dx_{i_1} \cdots dx_{i_r} dx_{j_1} \cdots dx_{j_r} = (-1)^r dx_{j_1} \cdots dx_{j_r} dx_{i_1} \cdots dx_{i_r}$. This gives immediately that the first statement is true.

Clearly when $\omega = dx_{i_1} \cdots dx_{i_r}$, then $\omega^2 = 0$ regardless of the parity of r. But if $\omega = dx_1 dx_2 + dx_3 dx_4$, then $\omega^2 = 2dx_1 dx_2 dx_3 dx_4 \neq 0$. Hence the second statement is false.