Date: March 3, Wednesday
Time: 08:40-10:30
Ali Sinan Sertöz

NAME:...

STUDENT NO:...

Math 214 Advanced Calculus – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let I be a non-empty closed and bounded interval in $\mathbb{R}$. Suppose that for every $x \in I$ there exists a non-negative C^∞ function f_x such that $f_x(x) > 0$ and $f_x'(t) = 0$ for all $t \notin I$. Show that there exists a non-negative C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) > 0$ for all $t \in I$ and $f'(t) = 0$ for all $t \notin I$.

Solution:
For each $x \in I$, since f is continuous and positive at x, there exists an open neighborhood U_x of x such that $f(t) > 0$ for all $t \in U_x$. Since $I \subset \bigcup_{x \in I} U_x$ and I is compact, there exists a finite number of points $x_1, \ldots, x_m \in I$ such that $I \subset \bigcup_{i=1}^m U_{x_i}$. Define $f = f_{x_1} + \cdots + f_{x_m}$. Check that this satisfies the requirements.

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Q-2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and E be a non-empty subset of $\mathbb{R}^n$.

(a): Mark each of the following statements as TRUE or FALSE.
Grading: Each correct answer is 2 points, each wrong answer is -3 points. No answer is 0 points.
(i) If E is open, then $f(E)$ is also open. **FALSE**
$f(x) = x^2$, $E = (-1, 1)$, $f(E) = [0, 1)$.
(ii) If E is closed, then $f(E)$ is also closed. **FALSE**
$f(x) = 1/x$, $E = [1, \infty)$, $f(E) = (0, 1]$.
(iii) If E is compact, then $f(E)$ is also compact. **TRUE**
(iv) If E is bounded, then $f(E)$ is also bounded. **FALSE**
$f(x) = 1/x$, $E = (0, 1)$, $f(E) = (1, \infty)$.
(v) If E is connected, then $f(E)$ is also connected. **TRUE**
(b): Prove or disprove: If V is open in $\mathbb{R}^m$, then $f^{-1}(V)$ is open in $\mathbb{R}^n$.
(10 points)

Solution:
Let $x \in f^{-1}(V)$. Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset V$. By continuity of f at x, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}^n$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. In other words $f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset V$, or equivalently $B_\delta(x) \subset f^{-1}(V)$ proving that $f^{-1}(V)$ is open.

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Q-3) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and E be a non-empty subset of $\mathbb{R}^n$. Prove that if E is compact, then f is uniformly continuous on E.

Solution:
Let $\epsilon > 0$ be chosen. By continuity, for each $x \in E$, there exists a $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subset B_{\epsilon/2}(f(x))$. Since $E \subset \bigcup_{x\in E} B_{\delta_x}(x)$ and since E is compact, there exist a finite set of points $x_1, \ldots, x_k \in E$ such that $E \subset U_1 \cup \cdots \cup U_k$, where $U_i = B_{\delta_{x_i}}(x_i)$. Choose a $\delta > 0$ such that $0 < 2\delta < \min\{\delta_{x_1}, \ldots, \delta_{x_k}\}$.

Now for any $x \in E$, $x \in U_j$ for some j. Take any y with $|x-y| < \delta$, then $|y-x_j| \leq |y-x|+|x-x_j| < 2\delta < \delta_{x_j}$. Hence $x, y \in U_j$ and $|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$. This proves that f is uniformly continuous on E.

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Q-4) Observe that

$$\int_0^{\pi/2} \lim_{k\to\infty} \sqrt{\frac{2k+7}{5k-x}}\,\sin x\,dx = \int_0^{\pi/2} \sqrt{\frac{2}{5}}\,\sin x\,dx = \sqrt{\frac{2}{5}}.$$

We want to evaluate

$$\lim_{k\to\infty} \int_0^{\pi/2} \sqrt{\frac{2k+7}{5k-x}}\,\sin x\,dx.$$

(a): Under which conditions can we take the limit to the inside of the integral sign, in general?

(b): Are those conditions satisfied for this case?

Solution:

If I is compact, $f_k(x)$ pointwise increases (or pointwise decreases) to a function f on I as k goes to infinity, where each f_k and f are continuous, then

$$\lim_{k\to\infty} \int_I f_k(x)\,dx = \int_I f(x)\,dx,$$

which is Dini's theorem. We can easily check that the derivative of the integrant in the above integral with respect to k is $-1/2\ (2\,x+35)\,\frac{1}{\sqrt{\frac{2\,k+7}{5\,k-x}}}\,(5\,k-x)^{-2}$ at each point. Hence the functions decrease pointwise as k increases, and we can use Dini's theorem here.

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Q-5) For any function $f : \mathbb{R} \to \mathbb{R}$, define the oscillation $\omega_f(t)$ of f at t. Calculate $\omega_f(t)$ for the function f that is defined as follows:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution:

$$\omega_f(t) = \lim_{h \to 0+} \sup_{x,y \in (t-h,t+h)} (f(x) - f(y)).$$

We know that $\omega_f(t) = 0$ if f is continuous at t. Hence we need to calculate only $\omega_f(0)$.

For any $h > 0$, supremum and infimum of f on $(-h, h)$ are 1 and -1 respectively since f oscillates between 1 and -1 infinitely many times on any open interval around 0.

Hence $\omega_f(0) = 2$.