

Date: April 14, 2010, Wednesday

NAME:.....

Time: 08:40-10:30

Ali Sinan Sertöz

STUDENT NO:.....

**Math 214 Advanced Calculus – Midterm Exam II – Solutions**

1	2	3	4	5	TOTAL
20	20	20	20	20	100

*Please do not write anything inside the above boxes!*

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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**Q-1)** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function.

**(a):** Define what it means for  $f$  to be differentiable at  $p \in \mathbb{R}^n$ .

**(b):** Mark each of the following statements as TRUE or FALSE.

*Grading: Each correct answer is 2 points, each wrong answer is -3 points. No answer is 0 points.*

(i) If  $f$  is differentiable at  $p$ , then  $f$  is continuous at  $p$ .

(ii) If  $f$  is differentiable at  $p$ , then all first order partial derivatives of  $f$  exist at  $p$ .

(iii) If all first order partial derivatives of  $f$  exist at  $p$ , then  $f$  is differentiable at  $p$ .

(iv) If all second order partial derivatives of  $f$  exist at  $p$ , then  $f$  is differentiable at  $p$ .

(v) If  $f$  is differentiable at  $p$ , then all first order partial derivatives of  $f$  exist and are continuous at  $p$ .

(vi) If all directional derivatives of  $f$  exist at  $p$ , then  $f$  is continuous at  $p$ .

**Solution:**

**(a):**  $f$  is differentiable at  $p$  if there is a linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p) - T(h)}{|h|} = 0,$$

where  $h \in \mathbb{R}^n$ .

**(b)** T-T-F-T-F-F

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**Q-2 (a):** State the implicit function theorem for a function  $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  at a point  $(a, b) \in \mathbb{R}^{n+m}$ , where  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}^m$ .

**(b):** Show that the system of equations

$$\begin{aligned}u^2 + v^2 + xyz &= 33 \\ \frac{u}{x} + \frac{v}{y} + \frac{4}{z^2} &= 7\end{aligned}$$

can be solved for  $u$  and  $v$  in terms of  $x$ ,  $y$  and  $z$  around the point  $(x, y, z) = (2, 1, 2)$  such that  $u(2, 1, 2) = 2$  and  $v(2, 1, 2) = 5$ .

**Solution:**

**(a):** Let  $f = (f_1, \dots, f_n)$  and let the coordinates of  $\mathbb{R}^{n+m}$  be given by  $(x_1, \dots, x_n, y_1, \dots, y_m)$ . Assume that  $f(a, b) = 0$ . If  $\det \left( \frac{\partial f_i}{\partial x_j}(a, b) \right)_{1 \leq i, j \leq n} \neq 0$ , then around  $b \in \mathbb{R}^m$  there exists a unique differentiable function  $g$  such that  $g(b) = a$  and  $f(g(t), t) = 0$  for all  $t$  in some neighbourhood of  $b$  in  $\mathbb{R}^m$ .

**(b):** Let  $f : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  be defined by

$$f(u, v, x, y, z) = (f_1, f_2) = (u^2 + v^2 + xyz - 33, \frac{u}{x} + \frac{v}{y} + \frac{4}{z^2} - 7).$$

Then

$$\det \left( \frac{\partial f_i}{\partial x_j}(2, 5, 2, 1, 2) \right) = \det \begin{pmatrix} 4 & 10 \\ 1/2 & 1 \end{pmatrix} = -1 \neq 0.$$

So  $u$  and  $v$  can be solved in terms of  $x$ ,  $y$  and  $z$  around  $(2, 1, 2)$ .

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**Q-3 (a):** State the inverse function theorem around the origin for a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**(b):** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined as

$$f(x, y) = (1 + x + 3y + x^3 + y^4, \cos x + 2 \sin y + \tan(x^2 + y^3)).$$

Show that  $f$  is invertible around  $(0, 0)$  and find  $D(f^{-1})(1, 1)$ , where  $D$  denotes the total derivative.

**Solution:**

**(a):** If  $\det Df(0) \neq 0$ , then  $f$  is invertible on some neighbourhood of the origin. Moreover,  $[Df^{-1}(f(0))] = [Df(0)]^{-1}$ .

**(b):**

$$Df(0, 0) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$$

This matrix is invertible, so  $f$  is locally invertible and since  $f(0, 0) = (1, 1)$ , we have

$$D(f^{-1})(1, 1) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3/2 \\ 0 & 1/2 \end{pmatrix}.$$

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**Q-4)** Prove or disprove that the following function is differentiable at the origin.

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

**Solution:**

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 1.$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0.$$

If  $f$  is differentiable at the origin, then its total derivative there must be  $(1, 0)$ .

Now we check the definition of differentiability of  $f$  at the origin.

$$\frac{f(x, y) - f(0, 0) - (1, 0) \cdot (x, y)}{(x^2 + y^2)^{1/2}} = \frac{-2xy^2}{(x^2 + y^2)^{3/2}} = \phi(x, y).$$

Since  $\phi(x, \lambda x) = \frac{-2\lambda^2}{(1 + \lambda^2)^{3/2}}$ , its limit as  $(x, y)$  goes to  $(0, 0)$  does not exist. Hence  $f$  is not differentiable at the origin.

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**Q-5)** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function.

**(a):** Let  $p$  be a point in  $\mathbb{R}^n$  and  $u$  be a unit vector in  $\mathbb{R}^n$ . Define the directional derivative of  $f$  in the direction of the vector  $u$  at the point  $p$ .

**(b):** Prove or disprove that if  $f$  is differentiable at  $p$ , then its directional derivatives in all directions exist at  $p$ .

**Solution:**

**(a):**  $D_u f(p) := \lim_{t \rightarrow 0} \frac{f(p + tu) - f(p)}{t}$  when the derivative exists.

**(b):** Assume that  $f$  is differentiable at  $p$ . Take  $h = tu$ . By definition we have

$$0 = \lim_{t \rightarrow 0} \left| \frac{f(p + tu) - f(p) - T(tu)}{t} \right| = \lim_{t \rightarrow 0} \left| \frac{f(p + tu) - f(p)}{t} - T(u) \right|.$$

Hence  $D_u f(p)$  exists and is equal to  $T(u)$ .