Math 302 Complex Calculus II
Homework 1 – Solutions
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**1:** Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^\alpha}$ where $\alpha > 1$. Take into account that $z^\alpha = exp(\alpha \ln z)$ is not defined at the origin.

**Solution:** We first find a branch of the function $z^\alpha$. Choose $\beta$ such that $2\pi/\alpha < \beta < 2\pi$. There exists such a $\beta$ since $\alpha > 1$. For $z = re^{i\theta}$, let us take the following branch of the logarithm

$$\log z = ln\, r + i\theta, \quad \beta - 2\pi < \theta < \beta. \tag{1}$$

(Here, $ln : (0, \infty) \to \mathbb{R}$ is the usual real logarithm).

Then, $z^\alpha = e^{\alpha \log z}$ is single-valued and analytic in $\mathcal{D} := \mathbb{C}\backslash\{z = re^{i\beta},\ 0 \le r < \infty\}$.

Let $\gamma$ be the following contour: $\gamma := \gamma_1 + \gamma_R - \gamma_2 - \gamma_\epsilon$, with

$$\begin{aligned}
\gamma_1 &= \{z = r :\ \epsilon \le r \le R\},\\
\gamma_R &= \{z = Re^{i\theta} :\ 0 \le \theta \le \frac{2\pi}{\alpha}\},\\
\gamma_2 &= \{z = re^{i\frac{2\pi}{\alpha}} :\ \epsilon \le r \le R\},\\
\gamma_\epsilon &= \{z = \epsilon e^{i\theta} :\ 0 \le \theta \le \frac{2\pi}{\alpha}\}.
\end{aligned}$$

Let

$$f(z) = \frac{1}{1+z^\alpha} = \frac{1}{1+e^{\alpha \log z}}, \qquad z \in \mathcal{D}.$$

We will integrate $f$ around the contour $\gamma$. Now, $f$ is analytic in $\mathcal{D}$ except at the points where $1 + z^\alpha = 0$. At these points,

$$z^\alpha = e^{\alpha \log z} = -1 = e^{(\pi+2k\pi)i}, \quad k \in \mathbb{Z}.$$

Since our branch of the logarithm is as in (1), the above equation is equivalent to (for $z = re^{i\theta}$)

$$e^{\alpha(lnr+i\theta)} = e^{(\pi+2k\pi)i}, \quad k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < 2\pi.$$

Solving the above equation for $r$ and $\theta$ we obtain

$$\alpha ln\, r = 0 \ \Leftrightarrow\ r = 1,$$

$$\alpha\theta = \pi + 2k\pi,\ k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < \beta \ \Leftrightarrow$$
$$\theta = \frac{\pi}{\alpha} + \frac{2k\pi}{\alpha},\ k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < \beta.$$

Of the above points, only $z = e^{i\pi/\alpha}$ lies inside of $\gamma$. Therefore, $f$ is analytic inside and on $\gamma$ except at the point $e^{i\pi/\alpha}$. Let us find the multiplicity of the pole of $f$ at $z = e^{i\pi/\alpha}$:

$$\frac{d}{dz}(z^\alpha + 1)\Big|_{z=e^{i\pi/\alpha}} = \alpha z^{\alpha-1}\Big|_{z=e^{i\pi/\alpha}} = \alpha e^{i\frac{\pi}{\alpha}(\alpha-1)} = -\alpha e^{-i\pi/\alpha} \neq 0.$$

Hence, $f$ has a simple pole at $z = e^{i\pi/\alpha}$ with residue $-1/(\alpha e^{-i\pi/\alpha})$. Applying Residue theorem, we obtain

$$\int_\gamma f(z)\ dz = 2\pi i \ \mathrm{Res}(f, e^{i\pi/\alpha}) = -\frac{2\pi i}{\alpha e^{-i\pi/\alpha}}. \tag{2}$$

On the other hand,

$$\int_\gamma f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\gamma_R} f(z)\ dz - \int_{\gamma_2} f(z)\ dz - \int_{\gamma_\epsilon} f(z)\ dz. \tag{3}$$

Let us evaluate each of the integrals above.

On $\gamma_1$: $\quad z = r; \quad dz = dr,\ \epsilon \leq r \leq R; \quad z^\alpha = r^\alpha,$

$$\int_{\gamma_1} f(z)\ dz = \int_\epsilon^R \frac{dr}{1 + r^\alpha}.$$

On $\gamma_R$: $\quad z = Re^{i\theta}; \quad dz = iRe^{i\theta}\ d\theta,\ 0 \leq \theta \leq 2\pi/\alpha;$

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(lnR + i\theta)} = R^\alpha e^{i\alpha\theta}.$$

$$\int_{\gamma_R} f(z)\ dz = \int_0^{2\pi/\alpha} \frac{iRe^{i\theta} d\theta}{1 + R^\alpha e^{i\alpha\theta}}.$$

For $R > 1$,

$$|1 + R^\alpha e^{i\alpha\theta}| \geq |R^\alpha e^{i\alpha\theta}| - 1 = R^\alpha - 1.$$

Therefore,

$$\left|\int_{\gamma_R} f(z)\ dz\right| \leq \int_0^{2\pi/\alpha} \frac{R}{R^\alpha - 1} d\theta = \frac{2\pi}{\alpha}\frac{R}{R^\alpha - 1}.$$

Since $\alpha > 1$,

$$\int_{\gamma_R} f(z)\ dz \to 0 \quad \text{as } R \to \infty.$$

On $\gamma_2$: $\quad z = re^{i2\pi/\alpha}; \quad dz = e^{i2\pi/\alpha} dr,\ \epsilon \leq r \leq R;$

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(lnr + i2\pi/\alpha)} = e^{\alpha lnr + i2\pi} = r^\alpha.$$

$$\int_{\gamma_2} f(z)\ dz = \int_\epsilon^R \frac{e^{i2\pi/\alpha} dr}{1 + r^\alpha} = e^{i2\pi/\alpha} \int_\epsilon^R \frac{dr}{1 + r^\alpha}.$$

On $\gamma_\epsilon$: $\quad z = \epsilon e^{i\theta}; \quad dz = i\epsilon e^{i\theta}\ d\theta,\ 0 \leq \theta \leq 2\pi/\alpha;$

$z^\alpha = e^{\alpha \log z} = e^{\alpha(ln\epsilon + i\theta)} = \epsilon^\alpha e^{i\alpha\theta}.$

$$\int_{\gamma_\epsilon} f(z)\ dz = \int_0^{2\pi/\alpha} \frac{i\epsilon e^{i\theta} d\theta}{1+\epsilon^\alpha e^{i\alpha\theta}}.$$

For $\epsilon < 1$, we have

$$|1+\epsilon^\alpha e^{i\alpha\theta}| \geq 1 - |\epsilon^\alpha e^{i\alpha\theta}| = 1-\epsilon^\alpha.$$

So,

$$\left|\int_{\gamma_\epsilon} f(z)\ dz\right| \leq \int_0^{2\pi/\alpha} \frac{\epsilon}{1-\epsilon^\alpha} d\theta = \frac{2\pi}{\alpha}\frac{\epsilon}{1-\epsilon^\alpha}.$$

As $\epsilon \to 0,\ \epsilon/(1-\epsilon^\alpha) \to 0$. Therefore,

$$\int_{\gamma_\epsilon} f(z)\ dz \to 0 \quad \text{as } \epsilon \to 0.$$

By (2) and (3),

$$\int_{\gamma_1} f(z)\ dz + \int_{\gamma_R} f(z)\ dz - \int_{\gamma_2} f(z)\ dz - \int_{\gamma_\epsilon} f(z)\ dz = \int_\gamma f(z)\ dz = -\frac{2\pi i}{\alpha e^{-i\pi/\alpha}}.$$

Taking limits as $\epsilon \to 0$ and $R \to \infty$, we get

$$\begin{aligned} \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_{\gamma_1} f + \int_{\gamma_R} f - \int_{\gamma_2} f - \int_{\gamma_\epsilon} f\right] &= -\frac{2\pi i}{\alpha e^{-i\pi/\alpha}} \\ \left(1-e^{i2\pi/\alpha}\right) \cdot \int_0^\infty \frac{dr}{1+r^\alpha} &= -\frac{2\pi i}{\alpha e^{-i\pi/\alpha}}. \end{aligned}$$

. We conclude,

$$\int_0^\infty \frac{dr}{1+r^\alpha} = -\frac{2\pi i}{\alpha}\frac{1}{(e^{-i\pi/\alpha} - e^{i\pi/\alpha})} = \frac{\pi/\alpha}{(e^{i\pi/\alpha}-e^{-i\pi/\alpha})/2i} = \frac{\pi/\alpha}{sin(\pi/\alpha)}.$$

**2:** Evaluate $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^n}$ where $n \geq 1$ is an integer.

**Solution:** Write

$$f(z) = \frac{1}{(1+z^2)^n}.$$

Let $\gamma$ be the following contour ($R > 1$): $\gamma = \gamma_1 + \gamma_R$, with

$$\begin{aligned} \gamma_1 &= \{z = r : \ -R \leq r \leq R\}, \\ \gamma_R &= \{z = Re^{i\theta} : \ 0 \leq \theta \leq \pi\}. \end{aligned}$$

We will evaluate $\int_\gamma f(z)dz$ in two ways.

$f$ is analytic in $\mathbb{C}$ except at the points where $(1+z^2)^n = 0$. The factorization $(1+z^2)^n = (z-i)^n(z+i)^n$ shows that $(1+z^2)^n$ has zeros at the points $z = i$ and $z = -i$ with multiplicity $n$. We conclude that $f$ is analytic inside and on $\gamma$ except at the point $z = i$, where it has a pole of multiplicity $n$. The residue of $f$ at $z = i$ is:

$$\begin{aligned} \mathrm{Res}(f, i) &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}((z-i)^n f(z))\Big|_{z=i} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(\frac{1}{(z+i)^n})\Big|_{z=i} \\ &= \frac{1}{(n-1)!}(-1)^{n-1}\frac{n\cdot(n+1)\ldots(2n-2)}{(z+i)^{2n-1}}\Big|_{z=i} \\ &= \frac{1}{(n-1)!}(-1)^{n-1}\frac{(2n-2)!}{(n-1)!}\frac{1}{(2i)^{2n-1}} \\ &= \frac{(-1)^{n-1}}{i^{2n-1}}\frac{(2n-2)!}{(n-1)!(n-1)!}\frac{1}{2^{2n-1}}. \end{aligned}$$

By Residue theorem,

$$\begin{aligned} \int_\gamma f(z)dz &= 2\pi i \ \mathrm{Res}(f, i) = \pi\frac{(-1)^{n-1}}{(i^2)^{n-1}}\frac{(2n-2)!}{(n-1)!(n-1)!}\frac{1}{2^{2n-2}} \\ &= \frac{\pi(2n-2)!}{(n-1)!(n-1)!2^{2n-2}}. \end{aligned}$$

On the other hand,

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_R} f(z)dz.$$

On $\gamma_1$: $\quad z = r; \quad dz = dr, \ -R \leq r \leq R.$

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{dr}{(1+r^2)^n}.$$

On $\gamma_R$: $\quad z = Re^{i\theta}; \quad dz = iRe^{i\theta}d\theta, \ 0 \leq r \leq \pi.$

$$\int_{\gamma_R} f(z)dz = \int_0^\pi \frac{iRe^{i\theta}}{(1+R^2e^{i2\theta})^n}d\theta.$$

For $R > 1$, $|1 + R^2 e^{i2\theta}| \geq |R^2 e^{i2\theta}| - 1 = R^2 - 1$. Therefore,

$$\left| \int_{\gamma_R} f(z)dz \right| \leq \int_0^{\pi} \frac{R}{(R^2-1)^n} = \frac{\pi R}{(R^2-1)^n} d\theta.$$

Since $n \geq 1$, $\int_{\gamma_R} f(z)dz \to 0$ as $R \to \infty$.

We conclude

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_R} f(z)dz = \int_{\gamma} f(z)dz = \frac{\pi(2n-2)!}{(n-1)!(n-1)!2^{2n-2}}.$$

Letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \frac{dr}{(1+r^2)^n} = \frac{\pi(2n-2)!}{(n-1)!(n-1)!2^{2n-2}}.$$

Therefore,

$$\int_0^{\infty} \frac{dr}{(1+r^2)^n} = \frac{\pi(2n-2)!}{(n-1)!(n-1)!2^{2n-1}}.$$

**3:** Find a conformal mapping of the disc $x^2 + (y-1)^2 < 1$ onto the first quadrant $x, y > 0$. Investigate the conformal property of your map also on the boundaries.

**Solution:** Let $f_1(z) = z - i$. Then $f_1$ maps the disc $\{z = x + iy : \ x^2 + (y-1)^2 < 1\}$ conformally onto the unit disc $\{z = x + iy : \ x^2 + y^2 < 1\}$.

Let $f_2(z) = (z-1)/(z+1)$. Then $f_2$ maps the unit disc conformally onto the left half plane $\{z = x + iy : \ x < 0\}$.

Let $f_3(z) = -iz$. Then $f_3$ maps the left half plane conformally onto the upper half plane $\{z = x + iy : \ y > 0\}$.

Let

$$f_4(z) = \sqrt{z} = e^{\frac{1}{2}\log z}, \qquad z \in \mathcal{D} := \mathbb{C}\backslash\{z = re^{-i\pi/2} : \ 0 \le r < \infty\},$$

where for $z \in \mathcal{D}, \ \log z = lnr + i\theta, \ -\pi/2 < \theta < 3\pi/2$. Then $f_4$ maps the upper half plane conformally onto the first quadrant $\{z = x + iy : \ x > 0, y > 0\}$.

Write

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z) = \sqrt{-i\frac{z-i-1}{z-i+1}},$$

where the meaning of ”square root” is as in explained above. Then $f$ maps the disc $\{z = x + iy : \ x^2 + (y-1)^2 < 1\}$ conformally onto the first quadrant.

$f$ is conformal at the boundary of the disc $\{z = x + iy : \ x^2 + (y-1)^2 < 1\}$, except at the points $z = -1 + i$ and $z = 1 + i$. It is clear that $f$ is undefined at the point $z = -1 + i$. At $z = 1 + i$, $f$ is also undefined since $f_4$ is undefined at $z = 0$.

**4:** Describe the image of the unit disc under the transformation $\ln\left(\frac{z-1}{z+1}\right)$, where an appropriate branch of the logarithm is used.

**Solution:** Let $f_1(z) = (z-1)/(z+1)$. Then $f_1$ maps the unit disc conformally onto the left half plane $\{z = x + iy : \ x < 0\}$. Take the following branch of the logarithm

$$\log z = lnr + i\theta, \quad 0 < \theta < 2\pi, \quad z = re^{i\theta}.$$

Then $\log z$ maps the left half plane conformally on the strip $\{z = x+iy : \ \pi/2 < y < 3\pi/2\}$.

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