1: Evaluate $\int_0^\infty \frac{dx}{1 + x^\alpha}$ where $\alpha > 1$. Take into account that $z^\alpha = \exp(\alpha \ln z)$ is not defined at the origin.

**Solution:** We first find a branch of the function $z^\alpha$. Choose $\beta$ such that $2\pi/\alpha < \beta < 2\pi$. There exists such a $\beta$ since $\alpha > 1$.

For $z = re^{i\theta}$, let us take the following branch of the logarithm

$$\log z = \ln r + i\theta, \quad \beta - 2\pi < \theta < \beta. \quad (1)$$

(Here, $\ln : (0, \infty) \to \mathbb{R}$ is the usual real logarithm).

Then, $z^\alpha = e^{\alpha \log z}$ is single-valued and analytic in $D := \mathbb{C}\{z = re^{i\beta}, \ 0 \leq r < \infty\}$.

Let $\gamma$ be the following contour: $\gamma := \gamma_1 + \gamma_R - \gamma_2 - \gamma_\epsilon$, with

$$\gamma_1 = \{z = r : \epsilon \leq r \leq R\},$$
$$\gamma_R = \{z = Re^{i\theta} : 0 \leq \theta \leq \frac{2\pi}{\alpha}\},$$
$$\gamma_2 = \{z = re^{i\frac{2\pi}{\alpha}} : \epsilon \leq r \leq R\},$$
$$\gamma_\epsilon = \{z = \epsilon e^{i\theta} : 0 \leq \theta \leq \frac{2\pi}{\alpha}\}.$$

Let

$$f(z) = \frac{1}{1 + z^\alpha} = \frac{1}{1 + e^{\alpha \log z}}, \quad z \in D.$$ 

We will integrate $f$ around the contour $\gamma$. Now, $f$ is analytic in $D$ except at the points where $1 + z^\alpha = 0$. At these points,

$$z^\alpha = e^{\alpha \log z} = -1 = e^{(\pi + 2k\pi)i}, \quad k \in \mathbb{Z}.$$

Since our branch of the logarithm is as in (1), the above equation is equivalent to (for $z = re^{i\theta}$)

$$e^{\alpha(ln r + i\theta)} = e^{(\pi + 2k\pi)i}, \quad k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < 2\pi.$$

Solving the above equation for $r$ and $\theta$ we obtain

$$\alpha ln r = 0 \iff r = 1,$$
$$\alpha \theta = \pi + 2k\pi, \quad k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < \beta \iff \theta = \frac{\pi}{\alpha} + \frac{2k\pi}{\alpha}, \quad k \in \mathbb{Z} \text{ and } \beta - 2\pi < \theta < \beta.$$
Of the above points, only $z = e^{i\pi/\alpha}$ lies inside of $\gamma$. Therefore, $f$ is analytic inside and on $\gamma$ except at the point $e^{i\pi/\alpha}$. Let us find the multiplicity of the pole of $f$ at $z = e^{i\pi/\alpha}$:

$$\frac{d}{dz}(z^\alpha + 1)\bigg|_{z=e^{i\pi/\alpha}} = \alpha z^{\alpha - 1}\bigg|_{z=e^{i\pi/\alpha}} = \alpha e^{i\pi(\alpha - 1)} = -\alpha e^{-i\pi/\alpha} \neq 0.$$  

Hence, $f$ has a simple pole at $z = e^{i\pi/\alpha}$ with residue $-\frac{1}{\alpha e^{-i\pi/\alpha}}$. Applying Residue theorem, we obtain

$$\int_{\gamma} f(z) \, dz = 2\pi i \text{Res}(f, e^{i\pi/\alpha}) = -\frac{2\pi i}{\alpha e^{-i\pi/\alpha}}. \quad (2)$$

On the other hand,

$$\int_{\gamma} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_R} f(z) \, dz - \int_{\gamma_2} f(z) \, dz - \int_{\gamma_\epsilon} f(z) \, dz. \quad (3)$$

Let us evaluate each of the integrals above.

On $\gamma_1$:  $z = r; \quad dz = dr, \quad \epsilon \leq r \leq R; \quad z^\alpha = r^\alpha$,

$$\int_{\gamma_1} f(z) \, dz = \int_{\epsilon}^{R} \frac{dr}{1 + r^\alpha}.$$

On $\gamma_R$:  $z = Re^{i\theta}; \quad dz = iRe^{i\theta} \, d\theta, \quad 0 \leq \theta \leq 2\pi/\alpha$;

$$z^\alpha = e^{\alpha \log z} = e^{\alpha (\ln R + i\theta)} = R^\alpha e^{i\alpha \theta}.$$  

$$\int_{\gamma_R} f(z) \, dz = \int_{0}^{2\pi/\alpha} \frac{iRe^{i\theta} \, d\theta}{1 + R^\alpha e^{i\alpha \theta}}.$$  

For $R > 1$,

$$|1 + R^\alpha e^{i\alpha \theta}| \geq |R^\alpha e^{i\alpha \theta}| - 1 = R^\alpha - 1.$$  

Therefore,

$$\left| \int_{\gamma_R} f(z) \, dz \right| \leq \int_{0}^{2\pi/\alpha} \frac{R}{R^\alpha - 1} \, d\theta = \frac{2\pi}{\alpha} \frac{R}{R^\alpha - 1}.$$  

Since $\alpha > 1$,

$$\int_{\gamma_R} f(z) \, dz \to 0 \quad \text{as} \quad R \to \infty.$$  

On $\gamma_2$:  $z = re^{i2\pi/\alpha}; \quad dz = e^{i2\pi/\alpha} \, dr, \quad \epsilon \leq r \leq R$;

$$z^\alpha = e^{\alpha \log z} = e^{\alpha (\ln r + i2\pi/\alpha)} = e^{\alpha \ln r + i2\pi} = r^\alpha.$$  

$$\int_{\gamma_2} f(z) \, dz = \int_{\epsilon}^{R} \frac{e^{i2\pi/\alpha} \, dr}{1 + r^\alpha} = e^{i2\pi/\alpha} \int_{\epsilon}^{R} \frac{dr}{1 + r^\alpha}.$$  

On $\gamma_\epsilon$:  $z = \epsilon e^{i\theta}; \quad dz = i\epsilon e^{i\theta} \, d\theta, \quad 0 \leq \theta \leq 2\pi/\alpha$;
\[ z^\alpha = e^{\alpha \log z} = e^{\alpha (\ln z + i\theta)} = e^{\alpha e^{i\alpha \theta}}. \]

\[ \int_{\gamma_\epsilon} f(z) \, dz = \int_0^{2\pi/\alpha} \frac{i\epsilon e^{i\theta} \, d\theta}{1 + e^{\alpha e^{i\alpha \theta}}}. \]

For \( \epsilon < 1 \), we have

\[ |1 + \epsilon^\alpha e^{i\alpha \theta}| \geq 1 - |\epsilon^\alpha e^{i\alpha \theta}| = 1 - \epsilon^\alpha. \]

So,

\[ \left| \int_{\gamma_\epsilon} f(z) \, dz \right| \leq \int_0^{2\pi/\alpha} \frac{\epsilon}{1 - \epsilon^\alpha} \, d\theta = \frac{2\pi}{\alpha} \frac{\epsilon}{1 - \epsilon^\alpha}. \]

As \( \epsilon \to 0 \), \( \epsilon/(1 - \epsilon^\alpha) \to 0 \). Therefore,

\[ \int_{\gamma_\epsilon} f(z) \, dz \to 0 \quad \text{as} \quad \epsilon \to 0. \]

By (2) and (3),

\[ \int_{\gamma_1} f(z) \, dz + \int_{\gamma_R} f(z) \, dz - \int_{\gamma_2} f(z) \, dz - \int_{\gamma_\epsilon} f(z) \, dz = \int_{\gamma} f(z) \, dz = -\frac{2\pi i}{\alpha e^{-\pi/\alpha}}. \]

Taking limits as \( \epsilon \to 0 \) and \( R \to \infty \), we get

\[ \lim_{R \to \infty} \left[ \int_{\gamma_1} f + \int_{\gamma_R} f - \int_{\gamma_2} f - \int_{\gamma_\epsilon} f \right] = -\frac{2\pi i}{\alpha e^{-\pi/\alpha}} \]

\[ (1 - e^{i2\pi/\alpha}) \cdot \int_0^\infty \frac{dr}{1 + r^\alpha} = -\frac{2\pi i}{\alpha e^{-\pi/\alpha}}. \]

We conclude,

\[ \int_0^\infty \frac{dr}{1 + r^\alpha} = -\frac{2\pi i}{\alpha} \frac{1}{(e^{-i\pi/\alpha} - e^{i\pi/\alpha})} = \frac{\pi/\alpha}{(e^{i\pi/\alpha} - e^{-i\pi/\alpha})/2i} = \frac{\pi/\alpha}{\sin(\pi/\alpha)}. \]
2: Evaluate \( \int_0^\infty \frac{dx}{(1+x^2)^n} \) where \( n \geq 1 \) is an integer.

Solution: Write

\[
f(z) = \frac{1}{(1+z^2)^n}.
\]

Let \( \gamma \) be the following contour (\( R > 1 \)): \( \gamma = \gamma_1 + \gamma_R \), with

\[
\gamma_1 = \{ z = r : -R \leq r \leq R \}, \\
\gamma_R = \{ z = Re^{i\theta} : 0 \leq \theta \leq \pi \}.
\]

We will evaluate \( \int_\gamma f(z)dz \) in two ways.

\( f \) is analytic in \( \mathbb{C} \) except at the points where \( (1+z^2)^n = 0 \). The factorization \( (1+z^2)^n = (z-i)^n(z+i)^n \) shows that \( (1+z^2)^n \) has zeros at the points \( z = i \) and \( z = -i \) with multiplicity \( n \). We conclude that \( f \) is analytic inside and on \( \gamma \) except at the point \( z = i \), where it has a pole of multiplicity \( n \). The residue of \( f \) at \( z = i \) is:

\[
\text{Res}(f,i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}((z-i)^n f(z)) \bigg|_{z=i} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{1}{(z+i)^n}\right) \bigg|_{z=i}
\]

\[
= \frac{1}{(n-1)!} (-1)^{n-1} \frac{n \cdot (n+1) \ldots (2n-2)}{(2n-2)!} \frac{1}{(z+i)^{2n-1}} \bigg|_{z=i} = \frac{1}{(n-1)!} (-1)^{n-1} \frac{(2n-2)!}{(2n-2)!} \frac{1}{(z+i)^{2n-1}} \bigg|_{z=i}
\]

\[
= \frac{1}{(n-1)!} (-1)^{n-1} \frac{(2n-2)!}{(2n-2)!} \frac{1}{(z+i)^{2n-1}} \bigg|_{z=i}
\]

By Residue theorem,

\[
\int_\gamma f(z)dz = 2\pi i \text{ Res}(f,i) = \frac{\pi}{(i^2)^{n-1}} \frac{(2n-2)!}{(n-1)! (n-1)! 2^{2n-2}}
\]

On the other hand,

\[
\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_R} f(z)dz.
\]

On \( \gamma_1 \): \( z = r ; \quad dz = dr , \quad -R \leq r \leq R \).

\[
\int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{dr}{(1+r^2)^n}.
\]

On \( \gamma_R \): \( z = Re^{i\theta} ; \quad dz = iRe^{i\theta}d\theta , \quad 0 \leq \theta \leq \pi \).

\[
\int_{\gamma_R} f(z)dz = \int_{0}^{\pi} \frac{iRe^{i\theta}}{(1+R^2e^{2i\theta})^n}d\theta.
\]
For $R > 1$, $|1 + R^2 e^{2i\theta}| \geq |R^2 e^{2i\theta} - 1 = R^2 - 1$. Therefore,

$$\left| \int_{\gamma_R} f(z) \, dz \right| \leq \int_0^\pi \frac{R}{(R^2 - 1)^n} = \frac{\pi R}{(R^2 - 1)^n} \, d\theta.$$ 

Since $n \geq 1$, $\int_{\gamma_R} f(z) \, dz \to 0$ as $R \to \infty$.

We conclude

$$\int_{\gamma_1} f(z) \, dz + \int_{\gamma_R} f(z) \, dz = \int_\gamma f(z) \, dz = \frac{\pi (2n - 2)!}{(n - 1)! (n - 1)! 2^{2n-2}}.$$ 

Letting $R \to \infty$, we obtain

$$\int_{-\infty}^\infty \frac{dr}{(1 + r^2)^n} = \frac{\pi (2n - 2)!}{(n - 1)! (n - 1)! 2^{2n-2}}.$$ 

Therefore,

$$\int_0^\infty \frac{dr}{(1 + r^2)^n} = \frac{\pi (2n - 2)!}{(n - 1)! (n - 1)! 2^{2n-1}}.$$
3: Find a conformal mapping of the disc $x^2 + (y - 1)^2 < 1$ onto the first quadrant $x, y > 0$. Investigate the conformal property of your map also on the boundaries.

**Solution:** Let $f_1(z) = z - i$. Then $f_1$ maps the disc $\{z = x + iy : x^2 + (y - 1)^2 < 1\}$ conformally onto the unit disc $\{z = x + iy : x^2 + y^2 < 1\}$.

Let $f_2(z) = (z - 1)/(z + 1)$. Then $f_2$ maps the unit disc conformally onto the left half plane $\{z = x + iy : x < 0\}$.

Let $f_3(z) = -iz$. Then $f_3$ maps the left half plane conformally onto the upper half plane $\{z = x + iy : y > 0\}$.

Let $f_4(z) = \sqrt{z} = e^{\frac{1}{2} \log z}$, $z \in \mathbb{D} := \mathbb{C} \setminus \{z = re^{i\pi/2} : 0 \leq r < \infty\}$, where for $z \in \mathbb{D}$, $\log z = \ln r + i\theta$, $-\pi/2 < \theta < 3\pi/2$. Then $f_4$ maps the upper half plane conformally onto the first quadrant $\{z = x + iy : x > 0, y > 0\}$.

Write

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z) = \sqrt{-i\frac{z - i - 1}{z - i + 1}},$$

where the meaning of "square root" is as in explained above. Then $f$ maps the disc $\{z = x + iy : x^2 + (y - 1)^2 < 1\}$ conformally onto the first quadrant.

$f$ is conformal at the boundary of the disc $\{z = x + iy : x^2 + (y - 1)^2 < 1\}$, except at the points $z = -1 + i$ and $z = 1 + i$. It is clear that $f$ is undefined at the point $z = -1 + i$. At $z = 1 + i$, $f$ is also undefined since $f_4$ is undefined at $z = 0$. 
4: Describe the image of the unit disc under the transformation \( \ln \left( \frac{z - 1}{z + 1} \right) \), where an appropriate branch of the logarithm is used.

Solution: Let \( f_1(z) = \frac{z - 1}{z + 1} \). Then \( f_1 \) maps the unit disc conformally onto the left half plane \( \{ z = x + iy : x < 0 \} \). Take the following branch of the logarithm

\[
\log z = \ln r + i\theta, \quad 0 < \theta < 2\pi, \quad z = re^{i\theta}.
\]

Then \( \log z \) maps the left half plane conformally on the strip \( \{ z = x + iy : \pi/2 < y < 3\pi/2 \} \).

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