Due on December 11, 2006, Monday, Class time. No late submissions!

MATH 302 Homework 3 – Solutions

1: Find an integral expression for $\Gamma'(z)$ for $Re\ z > 0$. Justify your steps.

Solution: Clearly the temptation is to take the derivative under the integral sign first and then check if it is valid. We use the following advanced calculus theorem:

Theorem: Let $f : (0,\infty) \times [c,d] \longrightarrow \mathbb{R}$ be continuous, where $c, d \in \mathbb{R}$ are fixed, and the improper integral

$$F(x) = \int_0^\infty f(t,x)\ dt$$

exists for all $x \in [c,d]$. Assume that $\dfrac{d}{dx} f(t,x)$ exists and is continuous on $(0,\infty) \times [c,d]$ and that

$$\phi(x) = \int_0^\infty \frac{d}{dx} f(t,x)\ dt$$

converges uniformly on $[c,d]$. Then $F(x)$ is differentiable on $[c,d]$ and $F'(x) = \phi(x)$ for all $x \in [c,d]$.

Using this you can show that the derivative of Gamma function under the integral sign converges uniformly on compacta for $\Re z > 0$, and this justifies the result.

2: Prove that $\Gamma(z) = \lim_{n\to\infty} \displaystyle\int_0^\infty t^{z-1}\left(1 - \frac{t}{n}\right)^n\ dt$, $Re\ z > 0$.

Solution: This can be attacked by uniform convergence theorems which allow interchange of limit and integral but an elementary approach by comparing e^{-t} with $\left(1 - \frac{t}{n}\right)^n$ works fine here. This is Exercise 6 on page 242 and the solution is on page 288.

3: Find the radius of convergence for $f(z) = \displaystyle\sum_{n=0}^{\infty} z^{n!}$ and show that its circle of convergence is a natural boundary.

Solution: The radius of convergence is clearly 1. Direct application of Theorem 18.5 on page 231 shows that the boundary is natural.

4: Show that $\Gamma(z+1) = z\Gamma(z)$ for all $z \in \mathbb{C}$ except for $z = -n$ where $n \in \mathbb{N}$.

Solution: Recall that we define $\Gamma(z)$ for $\Re z > -1$ by $\Gamma(z+1)/z$. Then for $\Re z > -1$ we have

$$\Gamma(z+1) = \frac{\Gamma(z+2)}{z+2} = \frac{(z+1)\Gamma(z+1)}{z+1} \cdot \frac{z}{z} = z \cdot \frac{\Gamma(z+1)}{z} = z\Gamma(z).$$

By induction we can extend this to all of $\mathbb{C}$.

5: Show that $\displaystyle\sum_{p:\ \text{prime}} \frac{1}{p}$ diverges.

Solution: This is exercise 8 on page 242 and the solution is on page 288. A naive approach would be to take the derivative of both sides of

$$\zeta(z) = \prod_{p:prime} \left(1 - \frac{1}{p^z}\right)^{-1}, \quad \Re z > 1,$$

and use the Taylor expansion of $\log(1-t)$ to find, after taking limits as $z \to 1$,

$$\log \sum \frac{1}{n} = \sum \frac{1}{p} + \text{some convergent series.}$$

Since the left hand side diverges, $\sum 1/p$ should also diverge.

Please send comments to sertoz@bilkent.edu.tr.