Due on December 20, 2006, Wednesday, Class time. No late submissions!

MATH 302 Homework 4 – Solutions

1: Show that $\dfrac{1}{\zeta(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$, for $\Re z > 1$, where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Use the product formula

$$\frac{1}{\zeta(z)} = \prod_p \left(1 - \frac{1}{p^z}\right), \quad \Re z > 1,$$

and recall the elementary fact that every integer is the product of primes. Now expanding the right hand side and carrying out the multiplication give the required formula.

2: Show that $\dfrac{\zeta'(z)}{\zeta(z)} = -\displaystyle\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^z}$, for $\Re z > 1$, where

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \in \mathbb{N}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Again use the above product formula for $\zeta(z)$. Taking first the logarithm and then differentiating gives the result.

3: Let $f(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$.

(i) Show that the series converges for $\Re z > 1$.

(ii) Show that $f(z) = \left(1 - \dfrac{1}{2^{z-1}}\right) \zeta(z)$ for $\Re z > 1$.

(iii) Show that $\displaystyle\lim_{z \to 1} f(z) = \ln 2$.

(iv) Show that $f(z)$ is an entire function.

Solution: This is Exercise 7 on page 242 and the solution is on page 288.

Please send comments to sertoz@bilkent.edu.tr.