Due on December 20, 2006, Wednesday, Class time. No late submissions!

MATH 302 Homework 4 – Solutions

1: Show that \( \frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \), for \( \Re z > 1 \), where

\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes}, \\
0 & \text{otherwise}.
\end{cases}
\]

Solution: Use the product formula

\[
\frac{1}{\zeta(z)} = \prod_p \left( 1 - \frac{1}{p^z} \right), \quad \Re z > 1,
\]

and recall the elementary fact that every integer is the product of primes. Now expanding the right hand side and carrying out the multiplication give the required formula.

2: Show that \( \frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^z} \), for \( \Re z > 1 \), where

\[
\Lambda(n) = \begin{cases} 
\ln p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \in \mathbb{N}^+, \\
0 & \text{otherwise}.
\end{cases}
\]

Solution: Again use the above product formula for \( \zeta(z) \). Taking first the logarithm and then differentiating gives the result.

3: Let \( f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} \).

(i) Show that the series converges for \( \Re z > 1 \).
(ii) Show that \( f(z) = \left( 1 - \frac{1}{2^{2z-1}} \right) \zeta(z) \) for \( \Re z > 1 \).
(iii) Show that \( \lim_{z \to 1} f(z) = \ln 2 \).
(iv) Show that \( f(z) \) is an entire function.

Solution: This is Exercise 7 on page 242 and the solution is on page 288.
Please send comments to sertoz@bilkent.edu.tr.