Q-1) Using the residue techniques evaluate the integral \( \int_{0}^{\infty} \frac{x}{(x+1)(x+2)(x+4)} \, dx \).

**Solution:** Using the technique described on pp 134-136, this integral is equal to

\[ -\sum_{z_k = -1, -2, -4} \text{Res} \left( \frac{z}{(z+1)(z+2)(z+4)} \log(z); z_k \right) . \]

The above residues are \(-\pi i/3, \log(2) + \pi i\) and \(-4/3 \log(2) - 2\pi i/3\) respectively. They add up to \(-1/3 \log(2)\) and hence the value of the integral is \((1/3) \log(2)\).

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Q-2) Show that \(e^z - z = 0\) has infinitely many zeros and that each zero is a simple root.

**Solution:** Using theorem 16.13 on page 212, if the equation has only finitely many zeros then \(e^z - z = P(z)e^{Q(z)}\) where \(P(z)\) and \(Q(z)\) are polynomials. Taking the second derivative of both sides gives first that \(Q(z) = z\) and consequently \(P(z) = 1\). But this leads to the absurd conclusion that \(-z = 0\) for all \(z\). So the equation must have infinitely many zeros.

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Q-3) Let \(z_1, z_2, \ldots, z_k, \ldots\) be all the roots of \(e^z = z\). Let \(C_N\) be the square in the plane centered at the origin with sides parallel to the axes and each of length \(2\pi N\). Assume that \(\lim_{N \to \infty} \int_{C_N} \frac{e^z - 1}{z^2(e^z - z)} \, dz = 0\). Find \(\sum_{k=1}^{\infty} \frac{1}{z_k^2}\).

**Solution:** Letting \(f = \frac{e^z - 1}{z^2(e^z - z)}\), the above limit also converges to

\[ 2\pi i \left( \sum_{k=1}^{\infty} \text{Res}(f; z_k) + \text{Res}(f; 0) \right) . \]

We find that \(\text{Res}(f; z_k) = 1/z_k^2\) and \(\text{Res}(f; 0) = 1\), yielding \(\sum_{k=1}^{\infty} \frac{1}{z_k^2} = -1\).

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Q-4) Find a \(C\)-harmonic function \(f(x, y)\) in the unit disk whose restriction to the boundary is \(x^2 + y^3\).

**Solution:** Using the real part of \(z^2\) and the imaginary part of \(z^3\), for simplicity, we find that

\[ f(x, y) = \frac{1}{2}(x^2 - y^2) + \frac{1}{4}(y^3 - 3x^2y) + \frac{3}{4}y + \frac{1}{2} . \]
Q-5) Throughout the semester we met two methods to calculate \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). Use one of these methods to find this sum.

**Solution:** The sum is \( \pi^2/6 \). One method is to use residue theory, and the other method is to use the infinite product representation of \( \sin \pi z \); see pages 141 and 223 respectively.