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Math 302 Complex Calculus – Final Exam – Solutions

Q-1) Using the residue techniques evaluate the integral $\displaystyle\int_0^\infty \frac{x}{(x+1)(x+2)(x+4)}\,dx.$

Solution: Using the technique described on pp 134-136, this integral is equal to

$$-\sum_{z_k=-1,-2,-4} Res\left(\frac{z}{(z+1)(z+2)(z+4)}\ \log(z); z_k\right).$$

The above residues are $-\pi i/3$, $\log(2)+\pi i$ and $-(4/3)\log(2)-2\pi i/3$ respectively. They add up to $-(1/3)\log(2)$ and hence the value of the integral is $(1/3)\log(2)$.

Q-2) Show that $e^z - z = 0$ has infinitely many zeros and that each zero is a simple root.

Solution: Using theorem 16.13 on page 212, if the equation has only finitely many zeros then $e^z - z = P(z)e^{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials. Taking the second derivative of both sides gives first that $Q(z)=z$ and consequently $P(z)=1$. But this leads to the absurd conclusion that $-z=0$ for all z. So the equation must have infinitely many zeros.

Q-3) Let $z_1, z_2, \ldots, z_k, \ldots$ be all the roots of $e^z = z$. Let C_N be the square in the plane centered at the origin with sides parallel to the axes and each of length $2\pi N$. Assume that $\displaystyle\lim_{N\to\infty}\int_{C_N}\frac{e^z-1}{z^2(e^z-z)}\,dz = 0$. Find $\displaystyle\sum_{k=1}^{\infty}\frac{1}{z_k^2}$.

Solution: Letting $f = \dfrac{e^z-1}{z^2(e^z-z)}$, the above limit also converges to

$2\pi i\left(\displaystyle\sum_{k=1}^{\infty} Res(f;z_k) + Res(f;0)\right)$. We find that $Res(f;z_k) = 1/z_k^2$ and $Res(f;0)=1$, yielding $\displaystyle\sum_{k=1}^{\infty}\frac{1}{z_k^2} = -1.$

Q-4) Find a C-harmonic function $f(x,y)$ in the unit disk whose restriction to the boundary is x^2+y^3.

Solution: Using the real part of z^2 and the imaginary part of z^3, for simplicity, we find that

$$f(x,y) = \frac{1}{2}(x^2-y^2) + \frac{1}{4}(y^3-3x^2y) + \frac{3}{4}y + \frac{1}{2}$$

is the required harmonic function.

Q-5) Throughout the semester we met two methods to calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Use one of these methods to find this sum.

Solution: The sum is $\pi^2/6$. One method is to use residue theory, and the other method is to use the infinite product representation of $\sin \pi z$; see pages 141 and 223 respectively.
