Math 302 Complex Calculus – Make-up Exam – Solutions

Q-1) Use residue theory to evaluate \( \int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \).

Solution: Consider the path from \(-R\) to \(R\) along the real line and then along the semicircle centered at the origin with radius \(R\) and lying in the upper half plane, with \(R > 2\). Integrate the function along this contour. On one hand the integral is \(2\pi i\) times the sum of the residues at \(z = i\) and \(z = 2i\), on the other hand the limit as \(R \to \infty\) is twice the above integral. This gives the value of the integral as \(\pi/6\).

Q-2) Find a conformal mapping of the first quadrant onto the unit disk mapping the points \(1 + i\) and \(0\) onto the points \(0\) and \(i\) respectively.

Solution: Take \(z\) and send it to \(w = z^2\) first so that the region becomes the upper half plane. Then using the general form of a conformal map from the upper half plane onto the unit disk we find that the required mapping is
\[
z \mapsto -i \frac{z^2 - 2i}{z^2 + 2i},
\]
see page 172.

Q-3) Find a \(C\)-harmonic function on the unit disc which restricts to \(y^4\) on the boundary.

Solution: Using the real parts of \(z^4\) and \(z^2\) we find that the required function is \((1/8) x^4 - (3/4) x^2 y^2 + (1/8) y^4 - (1/2) x^2 + (1/2) y^2 + (3/8)\).

Q-4) Let \(\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt\) for \(\Re z > 0\). Show that \(\Gamma(z)\) can be extended as a meromorphic function to the entire complex plane with simple poles at non-positive integers.

Solution: See pages 235-236.

Q-5) Show that the sum \(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots\), which converges absolutely for \(\Re z > 1\), represents an entire function.

Solution: This is Exercise 7 on page 242 with its solution on page 288.