Date: 10 January 2012, Tuesday
Time: 09:00-11:00
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Show that $\sum_{p \; prime} \frac{1}{p}$ diverges.

Solution:

We have the identity

$$\zeta(z) = \frac{1}{\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)} \quad \text{for} \quad \operatorname{Re} z > 1.$$

We also know that: If $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge, then $\prod_{k=1}^{\infty}(1 + z_k)$ converges. (This is an exercise from the book, and also was a midterm exam question.)

Since $\zeta(z)$ becomes infinite as z approaches 1, the infinite product $\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)$ diverges to zero.

Take z_k as -1 times the k-th prime.

Since the infinite product diverges and $\sum |z_k|^2$ converges, we must have $\sum z_k$ diverge according to the above fact.

This proves that $\sum_{p \; prime} \frac{1}{p}$ diverges.

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Q-2) Show that if $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge, then $\prod_{k=1}^{\infty}(1+z_k)$ converges.

Solution: *(This is Exercise 3 on page 226, solution on page 286, Second Edition.)*

The main result we use from complex analysis is that the convergence of $\prod_{k=1}^{\infty}(1+z_k)$ is equivalent to the convergence of $\sum_{k=1}^{\infty} \log(1+z_k)$. Therefore we will try to show the convergence of this infinite sum.

Since $\sum_{k=1}^{\infty} z_k$ converges, $|z_k| \le 1/2$ for all large k. So for all large k we have

$$
\begin{aligned}
|\log(1+z_k) - z_k| &= \left| -\frac{z_k^2}{2} - \frac{z_k^3}{3} - \cdots \right| \\
&\le |z_k|^2 \left(\frac{1}{2} + \frac{|z_k|}{3} + \cdots \right) \\
&\le |z_k|^2 \left(\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2^2 \cdot 4} + \cdots \right) \\
&< |z_k|^2 \left(\frac{1}{2} + \frac{1}{2^2} + \cdots \right) \\
&= |z_k|^2.
\end{aligned}
$$

By direct comparison from Calculus, $\sum_{k=1}^{\infty}(\log(1+z_k) - z_k)$ converges absolutely, since $\sum_{k=1}^{\infty} |z_k|^2$ converges.

Finally, as the difference of two convergent series

$$\sum_{k=1}^{\infty} \log(1+z_k) = \sum_{k=1}^{\infty}(\log(1+z_k) - z_k) - \sum_{k=1}^{\infty} z_k$$

converges, which is what we wanted to show.

NAME: STUDENT NO:

Q-3) Show that $\sum_{n=0}^{\infty} z^{n!}$ has the unit circle $|z| = 1$ as its natural boundary.

Solution: *(This is solved in class. It also follows directly from the statement of* Theorem 18.5 *on page 231.)*

Let ω be a k-th root of unity. Then $\omega^{n!} = 1$ for every $n \geq k$, so the infinite sum consists of infinitely many ones and diverges. Since the k-th roots of unity for $k = 1, 2, \ldots$ are dense on the unit circle, the series cannot be analytic on any open set containing any arc of the circle. Hence $|z| = 1$ is a natural boundary for the series.

Also note that from Theorem 18.5, $n_k = k!$ and $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty > 1$, so the series has its circle of convergence as a natural boundary. The circle of convergence, from Calculus, is $R = 1$.

NAME: STUDENT NO:

Q-4) Find a function $f(x,y)$ which is harmonic on $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on $\bar{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ such that $f(x,y) = x^3 + x^2 + x + 1$ on $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$.

Solution: *(This is a simplified version of* Example i *on page 207.)*

Let u be the real part of z^3. Then $u = x^3 - 3xy^2$ and is harmonic everywhere. Restricting u to ∂D we find $u|_{\partial D} = 4x^3 - 3x$, so

$$\frac{1}{4}u|_{\partial D} + \frac{7}{4}x = x^3 + x.$$

Let v be the real part of z^2. Then $v = x^2 - y^2$ and is harmonic everywhere. Restricting v to ∂D we find $v|_{\partial D} = 2x^1 - 1$, so

$$\frac{1}{2}v|_{\partial D} + \frac{3}{2} = x^2 + 1.$$

So we set

$$f(x,y) = \frac{1}{4}u + \frac{7}{4}x + \frac{1}{2}v + \frac{3}{2} = \frac{1}{4}x^3 - \frac{3}{4}xy^2 + \frac{7}{4}x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{3}{2}.$$

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Q-5) Let H be the upper half plane. Suppose that we have a function f analytic on H and continuous on $\bar{H}$, where $\bar{H}$ denotes the closure of H. Assume further that $|f(z)|$ is bounded on $\bar{H}$. Let $M = \sup\{|f(z)| \mid z \in \mathbb{R}\}$

Prove or disprove that $|f(z)| \leq M$ for all $z \in H$.

Solution:

We prove the statement.

If f is constant, there is nothing to prove. Assume then that f is not constant and hence $M > 0$.

Dividing f by M if necessary, we may assume without loss of generality that $M = 1$. Assume that K is an upper bound for $|f(z)|$ for $z \in \bar{H}$.

Fix any $z_0 \in H$. We claim that $|f(z_0)| \leq 1$.

For this purpose, consider the function

$$h(z) = \frac{f^n(z)}{z+i},$$

where n is a positive integer to be determined later. Clearly $|h(z)| \leq 1$ for all real z. Moreover for all $z \in H$ with $|z| = R > 1$, we have $|h(z)| \leq K^n/(R-1)$. Choose R large enough such that $K^n/(R-1) < 1$ and $R > |z_0|$. Consider the set

$$D_R = \{z \in H \mid |z| \leq R\}.$$

We showed above that $|h(z)| \leq 1$ on the boundary of $\bar{D}_R$, so by maximum modulus principle, $|h(z_0)| \leq 1$.

Hence for each $z_0 \in H$, we have

$$|h(z_0)| = \left|\frac{f^n(z_0)}{z_0+i}\right| \leq 1 \quad \text{or} \quad |f(z_0)| \leq |z_0+i|^{1/n}.$$

Taking n large enough, we can get

$$|f(z_0)| \leq 1 \quad \text{for all} \quad z_0 \in H,$$

which proves the claim and finishes the solution of the problem.