Due Date: October 31, 2011 Monday NAME:..

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Math 302 Complex Calculus II – Homework – Solution

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Please do not write anything inside the above boxes!

Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-5) Here we want to prove a finer version of a result in the book.
Let R be an open, simply connected, non-empty proper subset of the complex plane. Fix a point $z_0 \in R$. Define a collection of functions on R into the unit disk U as

$$\mathcal{F} = \{ f : R \to U \mid f \quad \text{is analytic and one-to-one,} \quad f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0 \, \}.$$

Show by a direct proof, i.e. not using the non-emptiness of a similar set of the textbook, that $\mathcal{F}$ is not empty.

Solution:

Fix a point a in the complement of R. Since R is simply connected and since $z - a$ is never zero, we can define a branch of logarithm on $z - a$ for all $z \in R$. Then define

$$g(z) = \exp(\frac{1}{2}\log(z-a)) = \sqrt{z-a}, \; z \in R.$$

g is one-to-one and analytic on R since $\log(z-a)$ is.

g is an open mapping, z_0 is an interior point of R, so $g(R)$ contains an open disk $D_\delta(g(z_0))$ of radius δ centered at $g(z_0)$ for some $\delta > 0$.

Since we fixed a branch of the square root function, $-g(z_0)$ is not achieved by g on R. Hence $-D_\delta(g(z_0))$ does not meet $g(R)$.

Note that

$$-D_\delta(g(z_0)) = D_\delta(-g(z_0)) = \{ w \in \mathbb{C} \mid |w + g(z_0)| < \delta \}.$$

For any $z \in R$, since $g(z) \notin -D_\delta(g(z_0))$, we must have

$$|g(z) + g(z_0)| \geq \delta; \quad \text{in particular} \quad 2|g(z_0)| \geq \delta.$$

Now define

$$f(z) = \frac{\delta}{4} \frac{|g'(z_0)|}{|g(z_0)|^2} \frac{g(z_0)}{g'(z_0)} \frac{g(z) - g(z_0)}{g(z) + g(z_0)}, \quad \text{for} \quad z \in R.$$

By the above arguments, we know that f is analytic on R. We claim that $f \in \mathcal{F}$.

First, check that since g is one-to-one, so is f.

Clearly $f(z_0) = 0$.

Also check directly that $f'(z_0) = \dfrac{\delta}{8} \dfrac{|g'(z_0)|}{|g(z_0)|^2} > 0$.

Finally, recalling that $|g(z) + g(z_0)| \geq \delta$ and $|g(z_0)| \geq \delta/2$, check that

$$\begin{aligned} \left| \frac{g(z) - g(z_0)}{g(z) + g(z_0)} \right| &= \left| 1 - \frac{2g(z_0)}{g(z) + g(z_0)} \right| \\ &= |g(z_0)| \left| \frac{1}{g(z_0)} - \frac{2}{g(z) + g(z_0)} \right| \\ &\leq |g(z_0)| \left(\frac{1}{|g(z_0)|} + \frac{2}{|g(z) + g(z_0)|} \right) \\ &\leq |g(z_0)| \left(\frac{2}{\delta} + \frac{2}{\delta} \right) \\ &= \frac{4|g(z_0)|}{\delta}. \end{aligned}$$

It is now clear that $|f(z)| \leq 1$ for all $z \in R$ and that $f \in \mathcal{F}$.