Q-5) Here we want to prove a finer version of a result in the book.

Let $R$ be an open, simply connected, non-empty proper subset of the complex plane. Fix a point $z_0 \in R$. Define a collection of functions on $R$ into the unit disk $U$ as

$$\mathcal{F} = \{ f : R \to U \mid f \text{ is analytic and one-to-one, } f(z_0) = 0 \text{ and } f'(z_0) > 0 \}.$$  

Show by a direct proof, i.e. not using the non-emptiness of a similar set of the textbook, that $\mathcal{F}$ is not empty.

Solution: