Due Date: December 26, 2011 Monday NAME:..

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Math 302 Complex Calculus II – Homework – Solution

6	7	8
10	10	10
10	10	10

Please do not write anything inside the above boxes!

Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-6) Find a formula for $\Gamma(\frac{n}{2})$, where n is a positive integer.

Solution:

Using the recursive formula $\Gamma(z+1) = z\Gamma(z)$ and induction, it can be easily shown that

$$\Gamma(\frac{2k+1}{2}) = \frac{1\cdot 3 \cdots (2k-1)}{2^k}\sqrt{\pi},$$

and

$$\Gamma(\frac{2k}{2}) = (k-1)!,$$

where k is a non-negative integer.

Q-7) Prove in detail and in your own words that

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt, \quad \text{for} \quad \operatorname{Re} z > 0.$$

Solution:

Fix a complex number $z = x + iy$ such that $\operatorname{Re} z = x > 0$.

Choose any $\epsilon > 0/$

Choose an integer N_1 such that for all $n \geq N_1$, $\left| \int_n^\infty t^{x-1} e^{-t}\, dt \right| < \epsilon/2$. This is possible since the integral for the Gamma function converges for $x > 0$.

Choose an integer N_2 such that for all $n \geq N_2$, $\frac{e}{2n}\Gamma(x+2) < \epsilon/2$.

In the following calculations, we take $n \geq N = \max\{N_1, N_2\}$ and $0 < t < n$.

From the Taylor expansion of $e^{-t/n}$ we find

$$0 \leq e^{-t/n} - \left(1 - \frac{t}{n}\right) \leq \frac{t^2}{2n^2}.$$

For $a > b > 0$, we recall that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}) \leq (a-b)na^{n-1}$. Using this with $a = e^{-t/n}$ and $b = (1 - t/n)$ together with the above inequality, we find

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq [e^{-t/n} - \left(1 - \frac{t}{n}\right)]ne^{(-t)+(t/n)} \leq \frac{e^{-t}t^2}{2n} e^{t/n} \leq \frac{e^{-t}t^2}{2n} e.$$

Finally, we have

$$\begin{aligned}
\left| \Gamma(z) - \int_0^n e^{z-1} \left(1 - \frac{t}{n}\right)^n dt \right| &\leq \int_0^n t^{x-1} \left| e^{-t} - \left(1 - \frac{t}{n}\right)^n \right| dt + \int_n^\infty t^{x-1} e^{-t}\, dt \\
&< \int_0^n t^{x-1} \frac{e^{-t}t^2}{2n} e\, dt + \epsilon/2 \\
&= \frac{e}{2n}\Gamma(x+2) + \epsilon/2 \\
&< \epsilon/2 + \epsilon/2 + \epsilon,
\end{aligned}$$

which proves that

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt, \quad \text{for} \quad \operatorname{Re} z > 0.$$

Q-8) Prove in detail and in your own words that $\sum_{p\ prime} \frac{1}{p}$ diverges.

Solution: We have the identity

$$\zeta(z) = \frac{1}{\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)} \quad \text{for} \quad \mathrm{Re}\, z > 1.$$

We also know that: If $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge, then $\prod_{k=1}^{\infty}(1 + z_k)$ converges. (This is an exercise from the book, and also was a midterm exam question.)

Since $\zeta(z)$ becomes infinite as z approaches 1, the infinite product $\prod_{p:prime}\left(1 - \frac{1}{p^z}\right)$ diverges to zero.

Take z_k as -1 times the k-th prime.

Since the infinite product diverges and $\sum |z_k|^2$ converges, we must have $\sum z_k$ diverge according to the above fact.

This proves that $\sum_{p\ prime} \frac{1}{p}$ diverges.