Date: 20 January 2012, Friday
Time: 09:00-11:00
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Makeup Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Find the infinite sum $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{9^n}$. Justify your steps.

Solution:

We first do some formal calculations and then justify them. Here C is a circle centered at the origin. To determine its radius is the key step in finding a justification for what we are doing.

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{9^n} &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{z^{n+1}} \, dz \right] \frac{1}{9^n} \\
&= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \left[\frac{(1+z)^2}{9z} \right]^n \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_C \frac{1}{1 - \frac{(1+z)^2}{9z}} \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_C \frac{9}{7z - 1 - z^2} \, dz.
\end{aligned}$$

The last integral will be the sum of the residues of the integrand, lying inside C. The roots of $7z - 1 - z^2 = 0$ are $z_1 = \frac{7 - 3\sqrt{5}}{2} \approx 0.14$ and $z_1 = \frac{7 + 3\sqrt{5}}{2} \approx 6.8$. The radius of C is determined by the step where we interchange the infinite sum and the integral. This step is justified if

$$\left| \frac{(1+z)^2}{9z} \right| < 1, \text{ or if } |z|^2 - 7|z| + 1 < 0.$$

The latter condition is satisfied if the radius of C is between $|z_1|$ and $|z_2|$. In this case the value of the integral is just the residue at z_1. Hence

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{9^n} = \mathrm{Res}\left(\frac{9}{7z - 1 - z^2}, z = z_1\right) = \frac{3}{\sqrt{5}} \approx 1.34.$$

NAME: STUDENT NO:

Q-2) Consider the integral $F(a,n)) = \int_I \frac{e^{-z}}{(z+1)^n}\, dz$ where I is the line $z(t) = a+it$ for $-\infty < t < \infty$. Here a is any real number, and n is any positive integer.

Evaluate $F(a,n)$. Justify your steps.

Solution:

First of all, we observe that the integral is not defined if $a = -a$. So assume that $a \neq 0$.

Let C_R be the right semicircle of radius $R > 0$ centered at $z = a$, and let γ_R be the closed contour whose one side is C_R and the other side is the vertical line on $z = a$. Then

$$\int_{a-iR}^{a+iR} \frac{e^{-z}}{(z+1)^n}\, dz + \int_{C_R} \frac{e^{-z}}{(z+1)^n}\, dz = \text{Res}(\frac{e^{-z}}{(z+1)^n}, z = \text{pole inside } \gamma_R).$$

As $R \to \infty$, the integral on C_R goes to zero, so

$$\int_I \frac{e^{-z}}{(z+1)^n}\, dz = \text{Res}(\frac{e^{-z}}{(z+1)^n}, z = \text{pole inside } \gamma_R).$$

If $a > -1$, there are no poles inside γ_R. If $a < -1$, then $z = -1$ is the pole inside γ_R, and the residue there is $\frac{(-1)^{n-1}e}{(n-1)!}$. Finally, we can write

$$F(a,n) = \begin{cases} 0, & \text{if } a > -1, \\ \frac{(-1)^{n-1}e}{(n-1)!}, & \text{if } a < -1, \\ \text{not defined}, & \text{if } a = -1. \end{cases}$$

NAME: STUDENT NO:

Q-3) The cross-ratio of four complex numbers z_1, z_2, z_3, z_4, denoted by (z_1, z_2, z_3, z_4), is the image of z_4 under the bilinear transformation which maps z_1, z_2, z_3 to $\infty, 0, 1$ respectively.

Show that the cross-ratio of four points is invariant under bilinear transformations, i.e. if T is any bilinear transformation then $(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4))$.

Solution: Let S be the bilinear map which sends z_1, z_2, z_3 into $\infty, 0, 1$ respectively. Then by definition $S(z_4) = (z_1, z_2, z_3, z_4)$. Observe that $S \circ T^{-1}$ sends Tz_1, Tz_2, Tz_3 into $\infty, 0, 1$ respectively. Then $S \circ T^{-1}(Tz_4) = (T(z_1), T(z_2), T(z_3), T(z_4))$. But $S \circ T^{-1}(Tz_4) = S(z_4) = (z_1, z_2, z_3, z_4)$, which is what we wanted to prove.

NAME: STUDENT NO:

Q-4) Find a function $f(x, y)$ which is harmonic on $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on $\bar{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ such that $f(x, y) = x^4 + x^3 - x^2 + x + 1$ on $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$.

Solution:

Let $u(x, y) = \mathrm{Re}(x + iy)^4 = x^4 - 6x^2y^2 + y^4$, and $v(x, y) = \mathrm{Re}(x + iy)^3 = x^3 - 3xy^2$. Being real parts of analytic functions, these are harmonic functions. Also $w(x, y) = (7/4)x + (7/8)$ is clearly harmonic. The required function is found to be

$$f(x, y) = \frac{1}{8}u(x, y) + \frac{1}{4}v(x, y) + w(x, y).$$

NAME: STUDENT NO:

Q-5) Show that the infinite product $\prod_{n=1}^{\infty}\left(1-\frac{(-1)^n}{n^2+n+2012}\right)$ converges.

Solution:

The general theory tells us that if $\sum_{n=1}^{\infty}|z_n|$ converges, then $\prod_{n=1}^{\infty}(1+z_n)$ converges.

Here we have $z_n = \frac{(-1)^n}{n^2+n+2012}$ which satisfies the premises of the general theory, so the given infinite product converges.