Date: 3 November 2011, Thursday
Time: 08:40-10:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Use the following at your own risk.

$$\begin{aligned}
\tan z &= \sum_{k=1}^{\infty} \frac{|B_{2k}|\, 2^{2k}(2^{2k}-1)}{(2k)!}\, z^{2k-1}, \ |z| < \pi/2. \\
\cot z &= \frac{1}{z} - \sum_{k=1}^{\infty} \frac{4^k\, |B_{2k}|}{(2k)!}\, z^{2k-1}, \ \ 0 < |z| < \pi. \\
\sec z &= \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!}\, z^{2k}, \ \ |z| < \pi/2. \\
\operatorname{cosec} z &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2)|B_{2k}|}{(2k)!}\, z^{2k-1}, \ \ 0 < |z| < \pi.
\end{aligned}$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30}.$$

$$E_0 = 1,\ E_1 = 0,\ E_2 = -1,\ E_3 = 0,\ E_4 = 5,\ E_5 = 0,\ E_6 = -61,\ E_7 = 0,\ E_8 = 1385.$$

$$\pi \coth(2\pi) = 3.14161456528446045677217 6...$$

NAME: STUDENT NO:

Q-1) Demonstrate the use of residue theory to find the value of the sum $\sum_{n=2}^{\infty} \frac{1}{n^2+4}$.

Solution:

From the theory of utilizing residue theory to infinite sums, we know that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+4} = -\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+4},\ z=2i\right) - \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+4},\ z=-2i\right).$$

It remains to calculate these residues. Since $\cot(\pi z)$ is analytic at $\pm 2i$, the residue calculation reduces to evaluation as follows:

$$\begin{aligned}
\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+4},\ z=2i\right) &= \left.\frac{\pi\cot(\pi z)}{z+2i}\right|_{z=2i} \\
&= -\frac{\pi\coth(2\pi)}{4} = -.7854036418..., \\
\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+4},\ z=-2i\right) &= \left.\frac{\pi\cot(\pi z)}{z-2i}\right|_{z=-2i} \\
&= -\frac{\pi\coth(2\pi)}{4} = -.7854036418....
\end{aligned}$$

Putting these into the above formula, we find

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+4} = \frac{\pi\coth(2\pi)}{2}.$$

Finally we have

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n^2+4} &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2+4} - \frac{1}{4} - \frac{2}{5}\right) \\
&= \frac{1}{2}\left(\frac{\pi\coth(2\pi)}{2} - \frac{1}{4} - \frac{2}{5}\right) \\
&= \frac{1}{2}\left(\frac{3.141614565}{2} - 0.250000000 - 0.40000000\right) \\
&= .4603836165....
\end{aligned}$$

NAME: STUDENT NO:

Q-2) For a real number α let $L(\alpha)$ be the line $z(t) = \alpha + it$ where $t \in \mathbb{R}$. Evaluate the integral

$$\int_{L(\alpha)} \frac{e^{-z}}{(z-5)^2}\, dz$$

for all possible values of α.

Solution:

If $\alpha = 5$, the integral is not defined since the integrand has a pole along the path.

Let $\alpha < 5$. Let I_R be the line $L(\alpha)$ for $-R \le t \le R$, and let C_R be the semicircle of radius R and centered at the point α on the real line, extending towards the right hand side. Choose R large enough so that the pole $z = 5$ is inside the closed contour $I_R + C_R$, traversed counterclockwise. Then

$$\begin{aligned} \int_{I_R + C_R} \frac{e^{-z}}{(z-5)^2}\, dz &= 2\pi i \,\mathrm{Res}\left(\frac{e^{-z}}{(z-5)^2}, z = 5\right) \\ &= 2\pi i \frac{-1}{e^5} \\ &\approx -0.042i. \end{aligned}$$

We further claim that the integral on C_R vanishes as R goes to infinity.For thsi note that for $|z - \alpha| = R$, we have

$$\left|\frac{e^{-z}}{(z-5)^2}\right| \le \frac{e^{-\alpha}}{(R + 5 - \alpha)^2},$$

and hence

$$\left|\int_{C_R} \frac{e^{-z}}{(z-5)^2}\, dz\right| \le (\pi R)\frac{e^{-\alpha}}{(R + 5 - \alpha)^2} \to 0 \quad \text{as } R \to \infty.$$

This shows that

$$\int_{L(\alpha)} \frac{e^{-z}}{(z-5)^2}\, dz = \lim_{R \to \infty} \int_{I_R + C_R} \frac{e^{-z}}{(z-5)^2}\, dz = -0.042i....$$

It is now clear that if $\alpha > 5$, then the integrand is analytic inside the closed contour $I_R + C_R$, and hence the integral is zero. The integral on C_R again vanishes as R goes to infinity.

Thus we find that

$$\int_{L(\alpha)} \frac{e^{-z}}{(z-5)^2}\, dz = \begin{cases} -\frac{2\pi i}{e^5} & \text{if } \alpha < 5, \\ \text{does not exists} & \text{if } \alpha = 5, \\ 0 & \text{if } \alpha > 5. \end{cases}$$

NAME: STUDENT NO:

Q-3) Classify all the automorphisms f of the unit disk with $f(0) = 0$ and $f'(0) > 0$.

Solution:

All automorphisms of the unit disk are of the form

$$f(z) = e^{i\theta} \left(\frac{z - \alpha}{1 - \bar{\alpha} z} \right)$$

for some real number θ and where clearly α, with $|\alpha| < 1$, is such that $f(\alpha) = 0$. In our case $\alpha = 0$, so the automorphism becomes

$$f(z) = e^{i\theta} z.$$

But now $f'(z) = e^{i\theta}$ and since $f'(0) > 0$, we must have $e^{i\theta} = 1$, forcing $f(z) = z$.

Hence the only such automorphism of the unit disk is the identity.

NAME: STUDENT NO:

Q-4) Let R be an open, non-empty subset of the complex plane. Assume that there exists a conformal mapping f of R onto the unit disk U. Choose any $z_0 \in R$. Prove or disprove that there exists a conformal mapping g from R onto U such that $g(z_0) = 0$ and $g'(z_0) > 0$.

Solution:

We prove the statement.

Let

$$g(z) = e^{i\theta}\left(\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}\right)$$

for some real θ. Here g is obtained by composing f with an automorphism of the unit disk sending $f(z_0)$ to zero.

We calculate and find that

$$g'(z_0) = \frac{f'(z_0)e^{i\theta}}{1 - |f(z_0)|^2}.$$

To make $g'(z_0) > 0$, all we need to do is to choose $\theta = -\operatorname{Arg} f'(z_0)$.

NAME: STUDENT NO:

Q-5) Let H be the upper half plane. Suppose that we have a function f analytic on H and continuous on $\bar{H}$, where $\bar{H}$ denotes the closure of H. Assume further that $|f(z)|$ is bounded on $\bar{H}$. Let $M = \sup\{|f(z)| \mid z \in \mathbb{R}\}$

Prove or disprove that $|f(z)| \leq M$ for all $z \in H$.

Solution:

We prove the statement.

If f is constant, there is nothing to prove. Assume then that f is not constant and hence $M > 0$.

Dividing f by M if necessary, we may assume without loss of generality that $M = 1$. Assume that K is an upper bound for $|f(z)|$ for $z \in \bar{H}$.

Fix any $z_0 \in H$. We claim that $|f(z_0)| \leq 1$.

For this purpose, consider the function

$$h(z) = \frac{f^n(z)}{z+i},$$

where n is a positive integer to be determined later. Clearly $|h(z)| \leq 1$ for all real z. Moreover for all $z \in H$ with $|z| = R > 1$, we have $|h(z)| \leq K^n/(R-1)$. Choose R large enough such that $K^n/(R-1) < 1$ and $R > |z_0|$. Consider the set

$$D_R = \{z \in H \mid |z| \leq R\}.$$

We showed above that $|h(z)| \leq 1$ on the boundary of $\bar{D}_R$, so by maximum modulus principle, $|h(z_0)| \leq 1$.

Hence for each $z_0 \in H$, we have

$$|h(z_0)| = \left|\frac{f^n(z_0)}{z_0+i}\right| \leq 1 \quad \text{or} \quad |f(z_0)| \leq |z_0+i|^{1/n}.$$

Taking n large enough, we can get

$$|f(z_0)| \leq 1 \quad \text{for all} \quad z_0 \in H,$$

which proves the claim and finishes the solution of the problem.