Date: 8 December 2011, Thursday
Time: 08:40-10:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. **A correct answer without proper reasoning may not get any credit.**

Q-1) Show that every non-constant meromorphic function on $\mathbb{C}$ is the ratio of two entire functions.

Solution: *(This is solved in class.)*

Let $\phi(z)$ be a meromorphic function whose poles are $\lambda_1, \lambda_2, \ldots$ repeated according to order. In other words, if z_0 is a pole of order 3, then z_0, z_0, z_0 is in the list. If the set of poles is finite, say $\lambda_1, \ldots, \lambda_n$, then consider the entire function $f(z) = (z - \lambda_1) \cdots (z - \lambda_n)$.

If the set of poles is infinite, since ϕ is non-constant, the set of poles has no accumulation point and hence diverges to infinity. According to Weierstrass Theorem (Theorem 17.7 on page 219, Second Edition) there is an entire function f vanishing exactly at the points $\lambda_1, \lambda_2, \ldots$.

Now that we have an entire function f vanishing on the poles of ϕ to multiplicity equal to the order of the pole, the function $g(z) = f(z)\phi(z)$ is an entire function vanishing on the zeros of ϕ to the same order as ϕ.

Then $\phi(z) = \dfrac{g(z)}{f(z)}$ as claimed.

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Q-2) Show that if $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge, then $\prod_{k=1}^{\infty}(1+z_k)$ converges.

Solution: *(This is Exercise 3 on page 226, solution on page 286, Second Edition.)*

The main result we use from complex analysis is that the convergence of $\prod_{k=1}^{\infty}(1+z_k)$ is equivalent to the convergence of $\sum_{k=1}^{\infty} \log(1+z_k)$. Therefore we will try to show the convergence of this infinite sum.

Since $\sum_{k=1}^{\infty} z_k$ converges, $|z_k| \le 1/2$ for all large k. So for all large k we have

$$\begin{aligned}
|\log(1+z_k) - z_k| &= \left| -\frac{z_k^2}{2} - \frac{z_k^3}{3} - \cdots \right| \\
&\le |z_k|^2 \left(\frac{1}{2} + \frac{|z_k|}{3} + \cdots \right) \\
&\le |z_k|^2 \left(\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 4} + \cdots \right) \\
&< |z_k|^2 \left(\frac{1}{2} + \frac{1}{2^2} + \cdots \right) \\
&= |z_k|^2.
\end{aligned}$$

By direct comparison from Calculus, $\sum_{k=1}^{\infty}(\log(1+z_k) - z_k)$ converges absolutely, since $\sum_{k=1}^{\infty} |z_k|^2$ converges.

Finally, as the difference of two convergent series

$$\sum_{k=1}^{\infty} \log(1+z_k) = \sum_{k=1}^{\infty}(\log(1+z_k) - z_k) - \sum_{k=1}^{\infty} z_k$$

converges, which is what we wanted to show.

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Q-3) Show that $f(z) = \prod_{k=0}^{\infty} \left(\frac{2(k-z)+1}{2k+1} \right) e^{(2z)/(2k+1)}$ is an entire function and determine all the solutions of $f(z) = 0$.

Solution: *(The solution is given in the* Note *immediately after Weierstrass Theorem on page 219.)*

Let $\lambda_k = k + \frac{1}{2}$. Then we observe that $\sum_{k=0}^{\infty} \frac{1}{\lambda_k}$ diverges but $\sum_{k=0}^{\infty} \frac{1}{\lambda_k^2}$ converges. So we can use

$$E_k(z) = \exp\left(\frac{z}{k+1/2} \right), \text{ for } k = 0, 1, \ldots$$

can be used as the convergence factor in Weierstrass product. Hence

$$f(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) E_k(z) = \prod_{k=0}^{\infty} \left(\frac{2k-1-2z}{2k+1} \right) e^{(2z)/(2k+1)}$$

is an entire function whose zero set is precisely the set of all λ_k for $k \geq 0$.

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Q-4) Find a function $f(x, y)$ which is harmonic on $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on $\bar{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ such that $f(x, y) = x^3 + x$ on $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$.

Solution: *(This is a simplified version of* Example i *on page 207.)*

Let u be the real part of z^3. Then $u = x^3 - 3xy^2$ and is harmonic everywhere. Restricting u to ∂D we find $u|_{\partial D} = 4x^3 - 3x$. We try to make this equal to f.

$$f(x, y)|_{\partial D} = x^3 + x = \frac{1}{4}u|_{\partial D} + \frac{7}{4}x.$$

So we set

$$f(x, y) = \frac{1}{4}u + \frac{7}{4}x = \frac{1}{4}\,x^3 - \frac{3}{4}\,xy^2 + \frac{7}{4}\,x.$$

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Q-5) Show that $\sum_{n=0}^{\infty} z^{n!}$ diverges at every point on the unit circle $|z| = 1$.

Postmortem note: The problem was intended to ask to show that $|z| = 1$ is a natural boundary. With the given wording, the problem became totally trivial. I will accept the trivial solution! What follows is the solution to the intended question.

Solution: *(This is solved in class. It also follows directly from the statement of* Theorem 18.5 *on page 231.)*

Let ω be a k-th root of unity. Then $\omega^{n!} = 1$ for every $n \geq k$, so the infinite sum consists of infinitely many ones and diverges. Since the k-th roots of unity for $k = 1, 2, \ldots$ are dense on the unit circle, the series cannot be analytic on any open set containing any arc of the circle. Hence $|z| = 1$ is a natural boundary for the series.

Also note that from Theorem 18.5, $n_k = k!$ and $\liminf_{k \to \infty} \frac{n_{k+1}}{n_k} = \infty > 1$, so the series has its circle of convergence as a natural boundary. The circle of convergence, from Calculus, is $R = 1$.