Date: 27 July 2011, Wednesday
Time: 10:00-12:00
Ali Sinan Sertöz

NAME:....................................................

STUDENT NO:....................................................

**Math 302 Complex Analysis II – Final Exam – Solutions**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Use the following at your own risk.

$$
\begin{aligned}
\tan z &= \sum_{k=1}^{\infty} \frac{|B_{2k}|\, 2^{2k}(2^{2k}-1)}{(2k)!}\, z^{2k-1}, \;\; |z| < \pi/2. \\
\cot z &= \frac{1}{z} - \sum_{k=1}^{\infty} \frac{4^k\,|B_{2k}|}{(2k)!}\, z^{2k-1}, \;\; 0 < |z| < \pi. \\
\sec z &= \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!}\, z^{2k}, \;\; |z| < \pi/2. \\
\operatorname{cosec} z &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2)|B_{2k}|}{(2k)!}\, z^{2k-1}, \;\; 0 < |z| < \pi.
\end{aligned}
$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30}.$$

$$E_0 = 1,\ E_1 = 0,\ E_2 = -1,\ E_3 = 0,\ E_4 = 5,\ E_5 = 0,\ E_6 = -61,\ E_7 = 0,\ E_8 = 1385.$$

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**Q-1)** Find the value of the following sum and write its exact value in the space provided. No partials!

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} =$$

**Solution:**

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} &= -\frac{1}{2} \operatorname{Res}\left(\frac{\pi \operatorname{cosec} \pi z}{z^4}; z=0\right) \\
&= -\frac{1}{2} \frac{(2^4-2)}{4!} |B_4| \pi^4 \\
&= -\frac{7\pi^4}{720}.
\end{aligned}$$

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**Q-2)** Evaluate the integral $\displaystyle\int_{351-i\infty}^{351+i\infty} \frac{2011^z}{(z+13)^{17}}\, dz$, where the principal branch of $\log$ is used in $2011^z$.

**Solution:**

Let $f(z) = 2011^z$. Then $\displaystyle\int_{351-i\infty}^{351+i\infty} \frac{f(z)}{(z+13)^{17}}\, dz = 2\pi i \operatorname{Res}\left(\frac{f(z)}{(z+13)^{17}}; z=-13\right) = 2\pi i\, \frac{f^{(16)}(-13)}{16!}$.

(For an explanation, see the first example on chapter 12.)

Since $f^{(16)}(-13) = (\log 2011)^{16}\, 2011^{-13}$, the answer is $\displaystyle 2\pi i\, \frac{(\log 2011)^{16}\, 2011^{-13}}{16!} \approx 4.2i \times 10^{-42}$.

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**Q-3)** Let $D$ be the unit disc around the origin. Find explicitly a function $f(x, y)$ such that $f$ is harmonic on $D$ and $f|_{\partial D} = x^4$. Can we arrange it so that $f(0, 0) = 1$? Explain why or how.

**Solution:**

Let $u(x, y) = \operatorname{Re} z^4 = x^4 - 6x^2y^2 + y^4$. Being the real part of an analytic function, $u(x, y)$ is harmonic on $D$.

On the unit circle we have $u|_{\partial D} = 8x^4 - 8x^2 + 1$.

We now have to find a harmonic function on $D$ which restricts to $x^2$ on $\partial D$. For this let $v(x, y) = \operatorname{Re} z^2 = x^2 - y^2$ which is harmonic being the real part of a holomorphic function.

On the unit circle we have $v|_{\partial D} = 2x^2 - 1$. Hence $x^2 = \dfrac{1}{2}v|_{\partial D} + \dfrac{1}{2}$. Set $g(x, y) = \dfrac{1}{2}v(x, y) + \dfrac{1}{2}$.

Returning to $x^4$, we observe that $x^4 = \dfrac{1}{8}u|_{\partial D} + g|_{\partial D} - \dfrac{1}{8}$.

Hence we can take $f(x, y) = \dfrac{1}{8}x^4 - \dfrac{3}{4}x^2y^2 + \dfrac{1}{8}y^4 + \dfrac{1}{2}x^2 - \dfrac{1}{2}y^2 + \dfrac{3}{8}$.

The general theory gives this as the unique function satisfying the requirements. Since $f(0, 0) = 3/8$, it cannot be changed.

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**Q-4)** Show that $f(z) = e^z - z$ has infinitely many zeros and that each zero is simple.

**Solution:**

Since $|f(z)| \leq e^{|z|^2}$ for large $z$, it is of finite order. If it has finitely many zeros, then it is of the form $Q(z)e^{P(z)}$ where $Q(z)$ and $P(z)$ are polynomials. We then have

$$e^{z-P(z)} - ze^{-P(z)} = Q(z).$$

But this is absurd since the LHS is clearly not a polynomial. Hence $f$ must have infinitely many zeros. Let $\alpha$ be a root of $f(z) = 0$. $f'(\alpha) = e^{\alpha} - 1 = 0$ holds only when $\alpha = 0$ but $0$ is not a root itself. So all roots must be simple.

Another way to see that the above equality is absurd is to check by induction that the $n$-th derivative of the LHS is of the form

$$e^{-P(z)} \left(R(z)e^z + S(z)\right)$$

where $R(z)$ and $S(z)$ are polynomials. When $n$ is larger than the degree of $Q(z)$, this expression must be identically equal to zero. This forces $e^z$ to be a rational function which is a contradiction. (A non-constant rational function has zeros and poles whereas $e^z$ has none!)

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**Q-5)** Recall that $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ for $\operatorname{Re} z > 0$. Choose any two real numbers $0 < x_0 < x_1$. Show that $|\Gamma(z)| \leq \Gamma(x_0) + \Gamma(x_1)$ for all $z$ with $x_0 \leq \operatorname{Re} z \leq x_1$.

**Solution:**

The proof is easy and can be given by the following self explanatory sequence of inequalities.

$$\begin{aligned}
|\Gamma(z)| &= \left| \int_0^\infty e^{-t} t^{z-1}\, dt \right| \\
&\leq \int_0^\infty e^{-t} t^{x-1}\, dt \\
&= \int_0^1 e^{-t} t^{x-1}\, dt + \int_1^\infty e^{-t} t^{x-1}\, dt \\
&\leq \int_0^1 e^{-t} t^{x_0-1}\, dt + \int_1^\infty e^{-t} t^{x_1-1}\, dt \\
&\leq \Gamma(x_0) + \Gamma(x_1).
\end{aligned}$$