Due Date: June 13, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 1 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let $f : U \to \mathbb{C}$ be a complex valued function of the form $f(z) = u(x, y) + iv(x, y)$, where U is an open region in $\mathbb{C}$. We know that if $f'(z)$ exists at every point $z \in U$, then the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold at every point of U.

What can you say about the converse of this fact?

Solution:

It is well known that Cauchy-Riemann equations alone do not imply complex differentiability. We need extra conditions.

Let $f(z) = u(x, y) + iv(x, y)$ where u and v are C^1 at $z_0 = x_0 + iy_0$. We will show that under this assumption the Cauchy-Riemann equations suffice for the existence of $f'(z_0)$.

Define:
$\Delta f = f(z_0 + \Delta z) - f(z_0)$, where $\Delta z = \Delta x + i\Delta y$.
$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$
$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$
Clearly, we have $\Delta f = \Delta u + i\Delta v$.

The increment theorem for C^1 functions gives:
$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ and
$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$ where
$\epsilon_k \to 0$ as $\Delta x, \Delta y \to 0$.

Using Cauchy-Riemann equations we can rewrite Δu and Δv as:
$\Delta u = u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$,
$\Delta v = v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$.

Finally we see that $\Delta f = \Delta u + i\Delta v = [u_x(x_0, y_0) + iv(x_0, y_0)]\Delta z + E$, where $E = \epsilon_1 \Delta x + \epsilon_2 \Delta y + i(\epsilon_3 \Delta x + \epsilon_4 \Delta y)$. Since $|\Delta x/\Delta z| < 1$ and $|\Delta y/\Delta z| < 1$, we see that $E/\Delta z \to 0$ as $\Delta z \to 0$. Hence

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = u_x + iv_x.$$

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Q-2) Find the Laurent expansion of $\operatorname{cosec} z$ around $z = 0$.

Solution:

We use Euler's identity $\dfrac{w}{e^w - 1} = \sum_{n=0}^{\infty} \dfrac{B_n}{n!} w^n$ as follows:

$$\begin{aligned}
\operatorname{cosec} z &= \frac{2i}{e^{iz} - e^{-iz}} = \frac{2ie^{iz}}{e^{2iz} - 1} \\
&= 2i\left(\frac{e^{iz} + 1 - 1}{(e^{iz} + 1)(e^{iz} - 1)}\right) \\
&= 2i\left(\frac{1}{e^{iz} - 1} - \frac{1}{e^{2iz} - 1}\right) \\
&= 2i\left(\frac{1}{iz} \cdot \frac{iz}{e^{iz} - 1} - \frac{1}{2iz} \cdot \frac{2iz}{e^{2iz} - 1}\right) \\
&= \frac{2}{z} \sum_{n=0}^{\infty} \frac{B_n}{n!} (iz)^n - \frac{1}{z} \sum_{n=0}^{\infty} \frac{B_n}{n!} (2iz)^n \\
&= \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(2 - 2^{2n})(-1)^n z^{2n-1} \\
&= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(2^{2n} - 2)|B_{2n}|}{(2n)!} z^{2n-1}, \ 0 < |z| < \pi,
\end{aligned}$$

where the radius of convergence can easily be determined by considering the next singularity of $\operatorname{cosec} z$.