Due Date: June 20, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 3 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Classify all invertible meromorphic functions from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$.

Solution:

Let f be such a meromorphic function. Since f is one-to-one, it has only one pole of order 1. If this pole is at infinity, then f is a polynomial, and since the pole is of order 1, f is a linear polynomial.

Now assume f has a pole at $z_0 \in \mathbb{C}$. Change coordinates by setting $Z = z - z_0$. Now $f(Z)$ has a pole at $Z = 0$ and hence cannot have another pole at infinity. Its value at infinity is determined by substituting $Z = 1/t$ and setting $t = 0$. Since f must be regular at infinity, its Laurent expansion around $Z = 0$ must be of the form $\dfrac{b_1}{Z} + a_0$ where $b_1 \neq 0$. Putting back $Z = z - z_0$ shows that f is of the form, $\dfrac{a_0 z + (b_1 - a_0 z_0)}{z - z_0} = \dfrac{az + b}{cz + d}$. Check that $ad - bc = -b_1 \neq 0$, so f is a Mobius transformation.

NAME: STUDENT NO:

Q-2) Let (z_1, z_2, z_3, z_4) and (z_1', z_2', z_3', z_4') be two four-tuples of distinct points with cross-ratios of λ and λ' respectively. Show that a Mobius transformation T exists with $T(z_i) = z_i'$, $i = 1, \ldots, 4$, if and only if $j(\lambda) = j(\lambda')$, where

$$j(\lambda) = 256\,\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

Solution:

Let T be the unique Mobius transformation sending z_1, z_2, z_3, z_4 to $\lambda, 1, 0, \infty$ in that order. Since $j(\lambda) = j(\lambda')$, we have $\lambda' \in \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)\}$. Then there exists a permutation $\sigma \in S_4$ and a unique Mobius transformation S such that S sends $z'_{\sigma(1)}, z'_{\sigma(2)}, z'_{\sigma(3)}, z'_{\sigma(4)}$ to $\lambda, 1, 0, \infty$ in that order. Then $S^{-1} \circ T(z_i) = z'_{\sigma(i)}$, $i = 1, 2, 3, 4$.